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Discrete-time Ruijsenaars-Schneider system and Lagrangian 1-form structure

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Abstract

We study the Lagrange formalism of the (rational) Ruijsenaars-Schneider (RS) system, both in discrete time as well as in continuous time, as a second example of a Lagrange 1-form structure in the sense of the recent paper [24]. The discrete-time model of the RS system was established some time ago arising via an Ansatz of a Lax pair, and was shown to lead to an exactly integrable correspondence (multivalued map)[15]. In this paper we present the full solution based on the commutativity of the discrete-time flows. The closure relation can be established by using the equations of motion. The connection with the KP systems is also studied. Performing successive continuum limits on the RS system, we establish the Lagrange 1-form structure for the corresponding continuum case of the RS model.

1 Introduction

The Ruijsenaars-Schneider (RS) system [20, 21], i.e., the relativistic version of the Calogero-Moser (CM) system, is integrable both in the classical and quantum regimes. The classical model was discovered in [20] by considering the Poincaré Poisson algebra associated with sine-Gordon solitons, and was motivated by the discovery in the late 1970s of explicit soliton-type S-matrices for some relativistic two-dimensional quantum field theories (such as the massive Thirring model, the quantum sine-Gordon theory and the $O(N)$ σ-models). For reasons elucidated below, we are interested in Lagrangian aspects of the RS model, which to our knowledge have never been considered. An obvious reason for this is that the Hamiltonian description corresponding to the system is no longer of Newtonian form, and hence the usual connection between the Hamiltonian and the Lagrangian description through the Legendre transformation seems no longer suitable. At the same time we are interested in the integrable time-discrete version of the RS system, which was proposed and studied in [15], where the Lagrangian description is more natural than the Hamiltonian one, because the finite-step time-iterate can be naturally viewed as a canonical transformation where the Lagrangian plays the role of its generating function. In [15] the corresponding discrete-time Lagrangian was found, but the continuum limits were not considered so far. As we shall show, the latter can be used to derive a natural Lagrangian description for the continuous RS model as well, but in the context of what we call a Lagrangian 1-form structure. We will now explain what we mean by this latter notion.

Recently, a novel point of view was developed on the role of the Lagrangian structure in integrable systems, cf. [7], where it was proposed that the fundamental property of *multidimensional consistency* can be made manifest in the Lagrangians by thinking of the latter as components of a difference (or differential) "Lagrange-form" when the flows are embedded in a multidimensional space-time. A new variational principle was formulated which involves not only variations with respect to the dependent variables of the theory, but also with respect to the geometry in the space of independent (discrete or continuous) variables. In [7], this was laid out in the case of two-dimensional lattice equations, whilst in [6] it was extended also to the case of the 3-dimensional bilinear KP equation (Hirota's equation). Furthermore, in [23] a universal Lagrangian structure was established for quadrilateral affine-linear lattice equations as well as for their corresponding continuous counterparts, the so-called *generating PDEs* of the system. The key property in all these systems, in which in a sense the integrability of the system resides, is that the Lagrangian form is closed on solutions of the equations of the motion (but not identically closed for arbitrary field values). This is believed to encode the multidimensional consistency of the system under consideration, but a formal proof of that connection remains to be given.

In the case of integrable systems of ODEs, like the equations of motion of integrable many-body systems, the Lagrangian form structure is that of Lagrange 1-forms. Recently, in collaboration with S Lobb, the authors studied a first example of such a Lagrange 1-form structure, namely the case of the discrete-time (rational) Calogero-Moser (CM) system, [24, 26]. The multidimensional consistency of the system in this case is represented by the co-existence of two or more independent *commuting* discrete-time flows in the case of three or more particles. Starting with the discrete-time case, we furthermore established the Lagrange structure also of the corresponding continuous case by performing systematic continuum limits on the discrete-time equations and Lagrangians. Of course, these systems exhibit also multi-Hamiltonian structure, in the sense that the classical Hamiltonians in involution with respect to a canonical Poisson structure, each of which generates its own time-flow in a corresponding time-variable, However, it is not the case that one can perform a naive Legendre transformation on each of these Hamiltonians separately to yield a proper Lagrangian structure that makes sense as a coherent system. In fact, the higher-order Lagrangians emerging from such a naive approach would yield rather complicated algebraic expressions which seem not to be suitable for further study. However, as we have shown in [24], a proper Lagrangian 1-form structure can be defined for the CM system, in which the components of the form are mixed, but polynomial in the time-derivatives, Lagrangians, which obey the crucial *closure property* expressing the commutativity of the flows *on solutions of the equations of the motion.* To derive these Lagrangians, the connection between the semi-discrete KP equation and the discrete-time CM system, which arises as the pole-reduction of the former, was instrumental in order to guide the proper choice of higher-order continuum limits obtained by systematic expansions performed on the discrete-time model, thus leading to the Lagrangians in the continuum case. Unfortunately, we do not know at this stage a Lagrangian of the semi-discrete KP equation in the form used, which would perhaps have allowed us to do the pole-reduction on the Lagrangians directly.

In the present paper, we proceed in the same flavor as with the paper [24], to establish the Lagrange 1-form structure of the discrete-time rational RS system. Whereas in the case of the discrete-time CM system there is a direct connection between the Lax matrices and the relevant Lagrangians, and where the mentioned closure relation is a direct consequence of the zero-curvature condition, such a direct connection is absent in the RS case. Nevertheless, a Lagrange structure for discrete-time RS system as was given in [15], and we show in the present paper that this Lagrangian structure (upon a small modification, and when extended into a space of multiple discrete times) still exhibits the closure property. Thus, this RS case seems to confirm that the Lagrangian 1-form structure is a general feature of integrable models in the 1-form case. Furthermore, whereas the discrete-time CM system arises as pole-reduction of a semi-discrete KP equation (with one continuous and two discrete independent variables) the discrete-time RS is connected by an analogous reduction to the fully discrete KP equation (with three discrete independent variables). Thus, the RS case seems to be quite a bit richer than the CM case of [24]. It is also interesting to note that one

of the lattice parameters of the KP equation (i.e., parameters which are associated with the grid size in each direction of the lattice) seems to play the role of the relativistic parameters of the RS model. In other words, the non-relativistic limit of the model corresponds to a continuum limit in one of the directions of the KP lattice.

Our focus in this paper is on the rational case of the RS model, even though some of our results on the Lagrangians can be extended straightforwardly to the trigonometric/hyperbolic case, because, as in the case of the CM system, we prefer all the statements we make to be backed up by the explicit solution of the equations of motion that can be constructed in this case. For instance, an important ingredient in the structure is what we call "constraint equations", which in addition to the one-dimensional equations of the motion can be verified explicitly for the solution obtained. These constraint equations involve the dynamics in two discrete variables (corresponding to trajectories in in the space of independent variables which involve corners or zig-zags). Since the starting point in the present paper is an Ansatz of a Lax pair, rather than a reduction from a KP system (the connection with the lattice KP is established *a posteriori*) the constraint equations emerge from zero-curvature conditions on the Lax matrices involving different time-directions. Thus, to establish that the whole system of discrete-time equations and constraints has a nontrivial family of solutions backs up the role the latter play in the structure.

The organization of the paper is as follows. In Section 2, we present the full solution of the discrete-time RS system, which requires commuting flows in different discrete-time directions. In Section 3, the Lagrangian 1-form structure of the discrete time RS system will be studied. The closure relation can be derived by direct computation involving the equations of motion as well as the constraints. In Section 4, the "skew" continuum limit will be derived guided by the exact solution of the discrete-time RS system system yielding what we call the semi-continuous RS system. In fact, the latter acts as a generating system for the continuum RS system, which allows us in Section 5 to derive the full limit, by which we recover the continuous-time RS hierarchy, together with the relevant continuous Lagrangians which depend in a mixed way on the various time-derivatives. It is the latter system of Lagrangians that constitutes the Lagrangian 1-form structure, and which exhibit the closure relation on the solutions of the system. In Section 6, the connection to KP systems is presented, by introducing a slight (parameter-)generalization of the original Lax representation. characteristic polynomial associated with the exact solution amounting to the corresponding lattice KP τ-function. Finally in Section 7 we briefly discuss the non-relativistic limit, demonstrating that in that limit the relevant lattice parameter effectively plays the role of the reciprocal of the speed of light. We end the paper with a discussion of some important points on the physical nature of the models studied here.

2 The discrete-time Ruijsenaars-Schneider system and commuting flows

In this Section we review the discrete-time RS system which has been introduced in [15]. This gives us an occasion to introduce appropriate notation which we will use throughout the paper. Furthermore, we derive the exact solution of the discrete equations of the motion and identify the constraint relations on commuting flows that can coexist in the system.

2.1 The single-flow RS system

Following [15] we start with an Ansatz for a Lax pair which in the rational cases takes the form:

$$\widetilde{\boldsymbol{\phi}} = \boldsymbol{M}_\kappa \boldsymbol{\phi} \ , \qquad \boldsymbol{L}_\kappa \boldsymbol{\phi} = \zeta \boldsymbol{\phi} \ , \tag{2.1a}$$

for a vector function ϕ and an eigenvalue ζ, in which the matrices $\boldsymbol{L}_\kappa$ and $\boldsymbol{M}_\kappa$ are given by

$$\boldsymbol{L}_\kappa = \frac{hh^T}{\kappa} + \boldsymbol{L}_0 \ , \tag{2.1b}$$

$$\boldsymbol{M}_\kappa = \frac{\widetilde{h}h^T}{\kappa} + \boldsymbol{M}_0 \ , \tag{2.1c}$$

where

$$\boldsymbol{L}_0 = \sum_{i,j=1}^{N} \frac{h_i h_j}{x_i - x_j + \lambda} E_{ij} \ , \quad \text{and} \quad \boldsymbol{M}_0 = \sum_{i,j=1}^{N} \frac{\widetilde{h}_i h_j}{\widetilde{x}_i - x_j + \lambda} E_{ij} \ . \tag{2.2}$$

In (2.1) the x_i are the particle positions, whilst the h_i are auxiliary variables which will be determined later. The *tilde* is a shorthand notation for the discrete-time shift, i.e. for $x_i(n,m) = x_i$, and we write $x_i(n+1,m) = \widetilde{x}_i$, and $x_i(n-1,m) = \underset{\sim}{x}_i$, where m is another discrete-time variable which will be specified later. The variable κ is the additional spectral parameter, whereas λ is a parameter of the system related to the non-relativistic limit. The matrix E_{ij} are the standard elementary matrices whose entries are given by $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

The compatibility condition of the system (2.1b):

$$\widetilde{\boldsymbol{L}}_\kappa \boldsymbol{M}_\kappa = \boldsymbol{M}_\kappa \boldsymbol{L}_\kappa \Rightarrow$$
$$\left(\frac{\widetilde{h}\widetilde{h}^T}{\kappa} + \widetilde{\boldsymbol{L}}_0\right)\left(\frac{\widetilde{h}h^T}{\kappa} + \boldsymbol{M}_0\right) = \left(\frac{\widetilde{h}h^T}{\kappa} + \boldsymbol{M}_0\right)\left(\frac{hh^T}{\kappa} + \boldsymbol{L}_0\right) \tag{2.3}$$

From the coefficients of $1/\kappa^2$ we derive the conservation law $tr\widetilde{\boldsymbol{L}}_\kappa = tr\boldsymbol{L}_\kappa$ leading to

$$\sum_{j=1}^{N} \widetilde{h}_j^2 = \sum_{j=1}^{N} h_j^2 \ , \tag{2.4}$$

and furthermore, the coefficients of $1/\kappa$ give

$$\widetilde{\boldsymbol{L}}_0 \widetilde{h}h^T + \widetilde{h}\widetilde{h}^T \boldsymbol{M}_0 = \boldsymbol{M}_0 hh^T + \widetilde{h}h^T \boldsymbol{L}_0 \ , \tag{2.5}$$

and the rest produces the equation

$$\widetilde{\boldsymbol{L}}_0 \boldsymbol{M}_0 = \boldsymbol{M}_0 \boldsymbol{L}_0 \ . \tag{2.6}$$

(2.5) and (2.6) produce the relations

$$\sum_{j=1}^{N}\left(\frac{\widetilde{h}_j^2}{\widetilde{x}_i - \widetilde{x}_j + \lambda} - \frac{h_j^2}{\widetilde{x}_i - x_j + \lambda}\right) = \sum_{j=1}^{N}\left(\frac{h_j^2}{x_j - x_l + \lambda} - \frac{\widetilde{h}_j^2}{\widetilde{x}_j - x_l + \lambda}\right) \ , \tag{2.7}$$

for all $i, j = 1, 2, ..., N$. Consequently, both sides of (2.7) must be independent of the external particle label. Thus, we find a coupled system of equations in terms of the variables h_i, and x_i in the form

$$\sum_{j=1}^{N}\left(\frac{\widetilde{h}_j^2}{\widetilde{x}_i - \widetilde{x}_j + \lambda} - \frac{h_j^2}{\widetilde{x}_i - x_j + \lambda}\right) = p \tag{2.8a}$$

$$\sum_{j=1}^{N}\left(\frac{h_j^2}{x_j - x_l + \lambda} - \frac{\widetilde{h}_j^2}{\widetilde{x}_j - x_l + \lambda}\right) = p \ , \tag{2.8b}$$

where p does not carry a particle label (but would still be a function of " n " and " m ").

Lemma 2.0.1. ***Lagrange interpolation formula**: Consider $2N$ noncoinciding complex numbers x_k and y_k, where $k = 1, 2, ..., N$. Then the following formula holds true:*

$$\prod_{k=1}^{N} \frac{(\xi - x_k)}{(\xi - y_k)} = 1 + \sum_{k=1}^{N} \frac{1}{(\xi - y_k)} \frac{\prod_{j=1}^{N}(y_k - x_j)}{\prod_{j=1, j\neq k}^{N}(y_k - y_j)} . \tag{2.9}$$

We now have

$$0 = 1 + \sum_{k=1}^{N} \frac{1}{(x_i - y_k)} \frac{\prod_{j=1}^{N}(y_k - x_j)}{\prod_{j=1, j\neq k}^{N}(y_k - y_j)} , \qquad i = 1, ..., N , \tag{2.10}$$

which is obtained by substituting $\xi = x_i$ into (2.9)*.*

In order to derive the closed set of equations of motion for the variables x_i we have to determine the variables h_i with the use of the Lagrange interpolation formula yielding

$$h_i^2 = -p \frac{\prod_{j=1}^{N}(x_i - x_j + \lambda)(x_i - \widetilde{x}_j - \lambda)}{\prod_{j\neq i}^{N}(x_i - x_j) \prod_{j=1}^{N}(x_i - \widetilde{x}_j)} , \tag{2.11a}$$

$$\widetilde{h}_i^2 = p \frac{\prod_{j=1}^{N}(\widetilde{x}_i - x_j + \lambda)(\widetilde{x}_i - \widetilde{x}_j - \lambda)}{\prod_{j\neq i}^{N}(\widetilde{x}_i - \widetilde{x}_j) \prod_{j=1}^{N}(\widetilde{x}_i - x_j)} , \tag{2.11b}$$

for $i = 1, 2, ..., N$ which we obtain the following system of equations

$$\frac{p}{\underset{\sim}{p}} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{(x_i - x_j + \lambda)}{(x_i - x_j - \lambda)} = \prod_{j=1}^{N} \frac{(x_i - \widetilde{x}_j)(x_i - \underset{\sim}{x}_j + \lambda)}{(x_i - \underset{\sim}{x}_j)(x_i - \widetilde{x}_j - \lambda)} . \tag{2.12}$$

(2.12) can be considered to be the product version of (2.11), is a system of N equations for $N+1$ unknowns, $x_1, ..., x_N$ and p. There is no equation for p separately, and thus it should be a priori given in order to get a closed set of equations. If p is a constant implying $p/\underset{\sim}{p}$ to be equal to unity, we obtain the equations of motion of what we would like to call the discrete-time RS system corresponding to the " $\widetilde{\ }$ " direction.

The exact solution of the equations of motion can be derived in a way similar to the continuous case of the rational RS model, cf. [15], cf. also [20, 19] using the explicit form of the Lax matrices (2.1b). The details are given in Appendix A, and we can summarize the conclusion as follows.

Proposition 2.1. *Let the $N \times N$ matrix function of the discrete variable n, $\boldsymbol{Y}(n)$, be given by*

$$\boldsymbol{Y}(n) = (p\boldsymbol{I} + \boldsymbol{\Lambda})^{-n} \left(\boldsymbol{Y}(0) - \frac{np\lambda}{p\boldsymbol{I} + \boldsymbol{\Lambda}} \right) (p\boldsymbol{I} + \boldsymbol{\Lambda})^{n} , \tag{2.13}$$

subject to the following condition on the initial value matrix

$$[\boldsymbol{Y}(0), \boldsymbol{\Lambda}] = \lambda\boldsymbol{\Lambda} + \text{rank } 1 . \tag{2.14}$$

then eigenvalues $x_i(n)$ of the matrix $\boldsymbol{Y}(n)$ coincide with the solutions for particle position of the discretetime RS system, i.e. they solve the discrete equations of motion (2.12)*.*

As a corollary, we have that equivalently the solutions can be found from the secular problem of the matrix:

$$\boldsymbol{Y}(0) - np\lambda(p\boldsymbol{I} + \boldsymbol{L}_0(0))^{-1} . \tag{2.15}$$

2.2 Commuting discrete flows

Following the construction in [24] we introduce another rational Lax pair in the form

$$\widehat{\boldsymbol{\phi}} = \boldsymbol{N}_\kappa \boldsymbol{\phi} \ , \qquad \boldsymbol{K}_\kappa \boldsymbol{\phi} = \zeta \boldsymbol{\phi} \ , \tag{2.16a}$$

for the same vector function $\boldsymbol{\phi}$, and

$$\boldsymbol{K}_\kappa = \frac{kk^T}{\kappa} + \boldsymbol{K}_0 \ , \tag{2.16b}$$

$$\boldsymbol{N}_\kappa = \frac{\widehat{k}k^T}{\kappa} + \boldsymbol{N}_0 \ , \tag{2.16c}$$

where

$$\boldsymbol{K}_0 = \sum_{i,j=1}^{N} \frac{k_i k_j}{x_i - x_j + \lambda} E_{ij} \ , \quad \text{and} \quad \boldsymbol{N}_0 = \sum_{i,j=1}^{N} \frac{\widehat{k}_i k_j}{\widetilde{x}_i - x_j + \lambda} E_{ij} \ . \tag{2.16d}$$

which provides the flow in an additional discrete-time variable m. Similar to the Lax pair in (2.1), in (7.2) the k_i are auxiliary variables which will be determined later. The *hat* is a shorthand notation for the discrete-time shift, i.e. for $x_i(n,m) = x_i$, and we write $x_i(n,m+1) = \widehat{x}_i$, and $x_i(n,m-1) = \underset{\frown}{x}_i$.

Obviously, the compatibility relations for (2.16) can be analysed in a very similar manner as to the ones for (2.1). Thus, we find a coupled system of equations in terms of the variables k_i, and x_i in the form

$$\sum_{j=1}^{N} \left(\frac{\widehat{k}_j^2}{\widehat{x}_i - \widehat{x}_j + \lambda} - \frac{k_j^2}{\widehat{x}_i - x_j + \lambda} \right) = q \tag{2.17a}$$

$$\sum_{j=1}^{N} \left(\frac{k_j^2}{x_j - x_l + \lambda} - \frac{\widehat{k}_j^2}{\widehat{x}_j - x_l + \lambda} \right) = q \ , \tag{2.17b}$$

where q does not carry a particle label. The system (2.17) can be resolved once again by using the Lagrange interpolation formula (2.10) yielding the resolution:

$$k_i^2 = -q \frac{\prod_{j=1}^{N}(x_i - x_j + \lambda)(x_i - \widehat{x}_j - \lambda)}{\prod_{j \neq i}^{N}(x_i - x_j) \prod_{j=1}^{N}(x_i - \widetilde{x}_j)} \ , \tag{2.18a}$$

$$\widehat{k}_i^2 = q \frac{\prod_{j=1}^{N}(\widehat{x}_i - x_j + \lambda)(\widehat{x}_i - \widehat{x}_j - \lambda)}{\prod_{j \neq i}^{N}(\widehat{x}_i - \widehat{x}_j) \prod_{j=1}^{N}(\widehat{x}_i - x_j)} \ , \tag{2.18b}$$

for $i = 1, 2, ..., N$, and from which we obtain the following system of equations

$$\frac{q}{\underset{\frown}{q}} \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{(x_i - x_j + \lambda)}{(x_i - x_j - \lambda)} = \prod_{j=1}^{N} \frac{(x_i - \widehat{x}_j)(x_i - \underset{\frown}{x}_j + \lambda)}{(x_i - \underset{\frown}{x}_j)(x_i - \widehat{x}_j - \lambda)} \ . \tag{2.19}$$

The product version (2.19) of (2.18), thus yields a system of N equations for $N+1$ unknowns, $x_1, ..., x_N$ and q. There is again no equation for q separately, and thus it should be a priori given in order to get a closed set of equations. If q is a constant implying $q/\underset{\frown}{q}$ to be equal to unity, we obtain the equations of motion of the discrete-time RS system corresponding to the " $\frown$ " direction in terms of the discrete-time variable m. Thus far, so similar.

Assuming now that the dependent variables depend simultaneously on both discrete time variables n and m, then to obtain a univalent solution of the equations of motion we must require that both flows, in the " $\sim$ " direction and the " $\frown$ " direction, commute. If so, then we can fix a value for n and solve the equations in the " $\frown$ " direction similarly as before, leading to the matrix $\boldsymbol{Y}$ which depends on n and m as follows:

$$\boldsymbol{Y}(n,m) = (q\boldsymbol{I} + \boldsymbol{\Lambda})^{-m} \left(\boldsymbol{Y}(n,0) - \frac{mq\lambda}{q\boldsymbol{I} + \boldsymbol{\Lambda}} \right) (q\boldsymbol{I} + \boldsymbol{\Lambda})^{m} \ , \tag{2.20}$$

subject to the constraint on the matrix $\boldsymbol{Y}$ at $m=0$:

$$[\boldsymbol{Y}(n,0),\boldsymbol{\Lambda}] = \lambda\boldsymbol{\Lambda} + \text{rank } 1 \ . \tag{2.21}$$

In order for this scenario to work there must be further constraints on the flows. This will lead to a system of "constraints" which can be readily obtained from the compatibility between Lax pairs (2.1c) and (2.16c). In fact, the compatibility relations between the simultaneous eigenvalue problems for the matrices $\boldsymbol{L}_\kappa$ and $\boldsymbol{K}_\kappa$ produces the following relations (see appendix B)

$$h_i^2 = \beta k_i^2 \ , \tag{2.22}$$

where

$$\beta = \frac{\sum_{j=1}^N h_j^2}{\sum_{j=1}^N k_j^2} \ . \tag{2.23}$$

Using (2.11) and (2.18), we obtain

$$\frac{p}{q\beta} = \prod_{j=1}^N \frac{(x_i - \widehat{x}_j - \lambda)(x_i - \widetilde{x}_j)}{(x_i - \widetilde{x}_j - \lambda)(x_i - \widehat{x}_j)} \ , \tag{2.24a}$$

$$\frac{p}{q\beta} = \prod_{j=1}^N \frac{(x_i - \underset{\frown}{x}_j + \lambda)(x_i - \underset{\sim}{x}_j)}{(x_i - \underset{\sim}{x}_j + \lambda)(x_i - \underset{\frown}{x}_j)} \ . \tag{2.24b}$$

We will refer to relations (2.24a) and (2.24b) as the *constraint equations* which guarantee the commutativity between the discrete-time flows with shifts " $\widetilde{}$ " and " $\widehat{}$ " in the variables n and m respectively.

Proposition: *The eigenvalues $x_1(n,m),\ldots,x_N(n,m)$ of the $N\times N$ matrix*

$$\begin{aligned}\boldsymbol{Y}(n,m) &= (p\boldsymbol{I}+\boldsymbol{\Lambda})^{-n}(q\boldsymbol{I}+\boldsymbol{\Lambda})^{-m}\boldsymbol{Y}(0,0)(p\boldsymbol{I}+\boldsymbol{\Lambda})^{n}(q\boldsymbol{I}+\boldsymbol{\Lambda})^{m} \\ &\quad -np\lambda(p\boldsymbol{I}+\boldsymbol{\Lambda})^{-1} - mq\lambda(q\boldsymbol{I}+\boldsymbol{\Lambda})^{-1}\end{aligned} \tag{2.25a}$$

in which the initial value matrix $\boldsymbol{Y}(0,0)$ is subject to the condition

$$[\boldsymbol{Y}(0,0)\,,\,\boldsymbol{\Lambda}] = \lambda\boldsymbol{\Lambda} + \text{rank}\,1 \ , \tag{2.25b}$$

obey both the discrete-time Ruijsenaars-Schneider systems given by eqs. (2.12) *and* (2.19) *as well as the systems of constraint equations given by* (2.24a) *and* (2.24b) .

In order to make a connection with an initial value problem, we mention that the initial value matrix $\boldsymbol{Y}(0,0)$ can be obtained from the diagonal matrix of initial values $\boldsymbol{X}(0,0)$ by a similarity transformation with a matrix $\boldsymbol{U}(0,0)$ which is an invertible matrix diagonalizing the initial Lax matrices $\boldsymbol{L}(0,0)$ and $\boldsymbol{K}(0,0)$. To find the latter, we need the initial values $x_i(0,0)$, $x_i(1,0)$ and $x_i(0,1)$, $(i=1,\ldots,N)$. We note that the secular problem can, hence, be reformulated as one for the following matrix

$$\mathfrak{X}(n,m) = \boldsymbol{X}(0,0) - np\lambda\boldsymbol{L}^{-1}(0,0) - mq\lambda\boldsymbol{K}^{-1}(0,0) \ , \tag{2.26}$$

and hence the solution is provided by the roots of the characteristic equation:

$$P_{\mathfrak{X}}(x) = \det(x\boldsymbol{I} - \mathfrak{X}(n,m)) = \prod_{i=1}^N (x - x_i(n,m)) \ . \tag{2.27}$$

Remark: We would like to mention at the end that if p and q are no longer constants the compatibility between (2.13) and (2.20) produces the conditions

$$\widehat{p}q = \widetilde{q}p \quad \text{and} \quad \widehat{p} + q = \widetilde{q} + p \ .$$

From these two equations, the only possible answers would be $p = p(n)$ and $q = q(m)$ implying that p and q must be functions of their own a discrete variable.

3 The Lagrangian 1-form and the closure relation

In the CM case [24], we obtained Lagrangians 1-form structure through the connection of the Lax representation. Here we also have the Lagrangian 1-form structure for the RS system, but the establishment is more difficult as connection through the Lax representation is no longer relevant. In this Section, we will first derive the Lagrangian 1-form for the discrete-time RS system and then establish the closure relation. We begin with rewriting (2.12) and (2.19) in the forms

$$\begin{aligned}\sum_{\substack{j=1\\ j\neq i}}^{N}\left(\ln(x_i-x_j+\lambda)-\ln(x_i-x_j-\lambda)\right) &= \sum_{j=1}^{N}\Big(\ln(x_i-\widetilde{x}_j)+\ln(x_i-\underset{\sim}{x}_j+\lambda)\\ &\quad -\ln(x_i-\underset{\sim}{x}_j)-\ln(x_i-\widetilde{x}_j-\lambda)\Big)\ , \quad (3.1)\\ \sum_{\substack{j=1\\ j\neq i}}^{N}\left(\ln(x_i-x_j+\lambda)-\ln(x_i-x_j-\lambda)\right) &= \sum_{j=1}^{N}\Big(\ln(x_i-\widehat{x}_j)+\ln(x_i-\underset{\sim}{x}_j+\lambda)\\ &\quad -\ln(x_i-\underset{\sim}{x}_j)-\ln(x_i-\widehat{x}_j-\lambda)\Big)\ . \quad (3.2)\end{aligned}$$

It is easy to show that the actions corresponding to these equations of motion are given by

$$S_{(n)} = \sum_{n}\mathscr{L}_{(n)}(\boldsymbol{x}(n),\boldsymbol{x}(n+1))\ , \tag{3.3}$$

$$S_{(m)} = \sum_{m}\mathscr{L}_{(m)}(\boldsymbol{x}(m),\boldsymbol{x}(m+1))\ , \tag{3.4}$$

where

$$\begin{aligned}\mathscr{L}_{(n)} &= \sum_{i,j=1}^{N}\left(f(x_i-\widetilde{x}_j)-f(x_i-\widetilde{x}_j-\lambda)\right)-\frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N}\Big(f(x_i-x_j+\lambda)\\ &\quad +f(\widetilde{x}_i-\widetilde{x}_j+\lambda)\Big)-\ln\left|\frac{p}{\sqrt{\beta}}\right|(\Xi-\widetilde{\Xi})\ , \quad (3.5)\\ \mathscr{L}_{(m)} &= \sum_{i,j=1}^{N}\left(f(x_i-\widehat{x}_j)-f(x_i-\widehat{x}_j-\lambda)\right)-\frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N}\Big(f(x_i-x_j+\lambda)\\ &\quad +f(\widetilde{x}_i-\widetilde{x}_j+\lambda)\Big)-\ln\left|q\sqrt{\beta}\right|(\Xi-\widehat{\Xi})\ , \quad (3.6)\end{aligned}$$

with $\Xi=\sum_{i=1}^{N}x_i$ the function $f(x)$ given by $f(x)=x\ln(x)$. The discrete-time Euler-Lagrange equations read

$$\frac{\partial\mathscr{L}_{(n)}}{\partial\widetilde{x}_i}+\frac{\widetilde{\partial\mathscr{L}_{(n)}}}{\partial x_i}=0\ ,\quad\text{and}\quad \frac{\partial\mathscr{L}_{(m)}}{\partial\widehat{x}_i}+\frac{\widehat{\partial\mathscr{L}_{(m)}}}{\partial x_i}=0\ ,$$

which lead to (3.1) and (3.2), respectively.

The additional terms in (3.5) and (3.6), containing the differences of the centre of mass, are needed in order to account for the constraint equations (2.24a) and (2.24b) as they arrive from the EL equations on discrete curves, which is a connected collection of line segments (i.e. elementary links on the lattice) with or without end points (i.e. closed or non-closed), involving corners (vertices connecting line segments with different directions).

Theorem 3.1. (The closure relation)

$$\widehat{\mathscr{L}_{(n)}(\boldsymbol{x},\widetilde{\boldsymbol{x}})}-\mathscr{L}_{(n)}(\boldsymbol{x},\widetilde{\boldsymbol{x}})-\widetilde{\mathscr{L}_{(m)}(\boldsymbol{x},\widehat{\boldsymbol{x}})}+\mathscr{L}_{(m)}(\boldsymbol{x},\widehat{\boldsymbol{x}})=0\ , \tag{3.7}$$

holds on the equations of motion and the constraint equations.

Proof. (3.7) can be written in the form

$$
\begin{aligned}
&\widehat{\mathscr{L}_{(n)}(\boldsymbol{x},\widetilde{\boldsymbol{x}})} - \mathscr{L}_{(n)}(\boldsymbol{x},\widetilde{\boldsymbol{x}}) - \widetilde{\mathscr{L}_{(m)}(\boldsymbol{x},\widehat{\boldsymbol{x}})} + \mathscr{L}_{(m)}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \\
&= \sum_{i,j=1}^{N} \widehat{x}_i \left(\ln \left| \frac{\widehat{x}_i - \widehat{\widetilde{x}}_j}{\widehat{x}_i - \widehat{\widetilde{x}}_j - \lambda} \frac{\widehat{x}_i - x_j - \lambda}{\widehat{x}_i - x_j} \right| - \ln \left| \frac{\widehat{x}_i - \widehat{x}_j + \lambda}{\widehat{x}_i - \widehat{x}_j - \lambda} \right| \right) \\
&- \sum_{i,j=1}^{N} \widetilde{x}_i \left(\ln \left| \frac{\widetilde{x}_i - \widehat{\widetilde{x}}_j}{\widetilde{x}_i - \widehat{\widetilde{x}}_j - \lambda} \frac{\widetilde{x}_i - x_j - \lambda}{\widetilde{x}_i - x_j} \right| - \ln \left| \frac{\widetilde{x}_i - \widetilde{x}_j + \lambda}{\widetilde{x}_i - \widetilde{x}_j - \lambda} \right| \right) \\
&+ \sum_{i,j=1}^{N} \left(\widehat{\widetilde{x}}_i \ln \left| \frac{\widetilde{x}_i - \widehat{\widetilde{x}}_j - \lambda}{\widetilde{x}_i - \widehat{\widetilde{x}}_j} \frac{\widetilde{x}_i - \widehat{\widetilde{x}}_j}{\widetilde{x}_i - \widehat{\widetilde{x}}_j - \lambda} \right| - x_i \ln \left| \frac{x_i - \widetilde{x}_j}{x_i - \widetilde{x}_j - \lambda} \frac{x_i - \widehat{x}_j - \lambda}{x_i - \widehat{x}_j} \right| \right) \\
&+ \left(\ln \left| q\sqrt{\beta} \right| - \ln \left| \frac{p}{\sqrt{\beta}} \right| \right) \left(\widetilde{\Xi} - \widehat{\widetilde{\Xi}} - \Xi + \widehat{\Xi} \right) \\
&+ \lambda \sum_{i,j=1}^{N} \left(\ln \left| \frac{\widehat{x}_i - \widehat{\widetilde{x}}_j - \lambda}{\widetilde{x}_i - \widehat{\widetilde{x}}_j - \lambda} \frac{x_i - \widehat{x}_j - \lambda}{x_i - \widetilde{x}_j - \lambda} \right| - \ln \left| \frac{\widehat{x}_i - \widehat{x}_j + \lambda}{\widetilde{x}_i - \widetilde{x}_j + \lambda} \right| \right) \ , \qquad (3.8)
\end{aligned}
$$

Using (2.12), (2.19), (2.24a) and (2.24b), we have

$$
\begin{aligned}
&\widehat{\mathscr{L}_{(n)}(\boldsymbol{x},\widetilde{\boldsymbol{x}})} - \mathscr{L}_{(n)}(\boldsymbol{x},\widetilde{\boldsymbol{x}}) - \widetilde{\mathscr{L}_{(m)}(\boldsymbol{x},\widehat{\boldsymbol{x}})} + \mathscr{L}_{(m)}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \\
&= \sum_{i=1}^{N} (\widehat{x}_i + \widetilde{x}_i - \widehat{\widetilde{x}}_i - x_i) \ln \left| \frac{p}{q\beta} \right| \\
&+ \left(\ln \left| q\sqrt{\beta} \right| - \ln \left| \frac{p}{\sqrt{\beta}} \right| \right) \left(\widetilde{\Xi} - \widehat{\widetilde{\Xi}} - \Xi + \widehat{\Xi} \right) \\
&+ \lambda \sum_{i,j=1}^{N} \left(\ln \left| \frac{\widehat{x}_i - \widehat{\widetilde{x}}_j - \lambda}{\widetilde{x}_i - \widehat{\widetilde{x}}_j - \lambda} \frac{x_i - \widehat{x}_j - \lambda}{x_i - \widetilde{x}_j - \lambda} \right| - \ln \left| \frac{\widehat{x}_i - \widehat{x}_j + \lambda}{\widetilde{x}_i - \widetilde{x}_j + \lambda} \right| \right) \ . \qquad (3.9)
\end{aligned}
$$

Using the fact that the last line of (3.9) vanishes on the exact solution (2.25a) and $\widetilde{\Xi} - \widehat{\widetilde{\Xi}} - \Xi + \widehat{\Xi} = 0$, then we have

$$
\widehat{\mathscr{L}_{(n)}(\boldsymbol{x},\widetilde{\boldsymbol{x}})} - \mathscr{L}_{(n)}(\boldsymbol{x},\widetilde{\boldsymbol{x}}) - \widetilde{\mathscr{L}_{(m)}(\boldsymbol{x},\widehat{\boldsymbol{x}})} + \mathscr{L}_{(m)}(\boldsymbol{x},\widehat{\boldsymbol{x}}) = 0 \ . \qquad (3.10)
$$

□

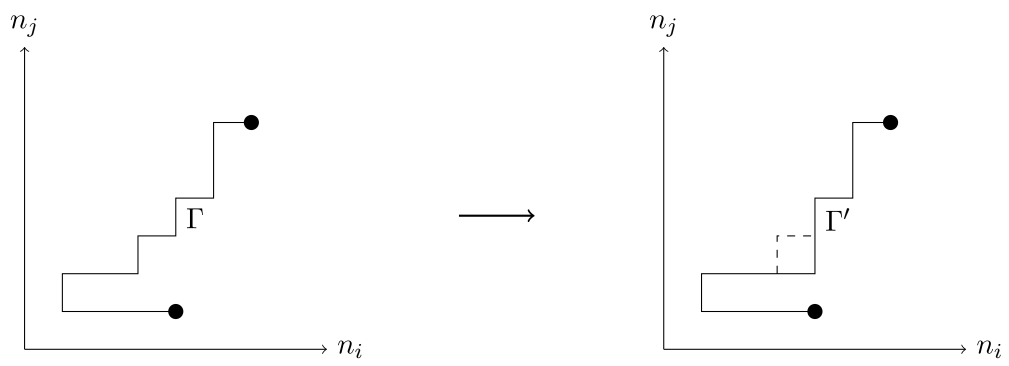

Figure 1: Deformation of the discrete curve Γ.

In [24] we described what we mean by the Lagrangian 1-form, but let us reiterate this here for the sake of self-contained of this paper. Let $\boldsymbol{e}_i$ represent the unit vector in the lattice

direction labeled by i and let any position in the lattice be identified by the vector $\boldsymbol{n}$, so that an elementary shift in the lattice can be created by the operation $\boldsymbol{n} \mapsto \boldsymbol{n} + \boldsymbol{e}_i$. Since the Lagrangian depends on $\boldsymbol{x}$ and its elementary shift in one discrete direction, it can be associated with an oriented vector $\boldsymbol{e}_i$ on a curve $\Gamma_i(\boldsymbol{n}) = (\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}_i)$, and we can treat these Lagrangians as defining a discrete 1-form $\mathscr{L}_i(\boldsymbol{n})$

$$\mathscr{L}_i(\boldsymbol{n}) = \mathscr{L}_i(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_i)), \tag{3.11}$$

which satisfies the following relation

$$\begin{aligned} &\mathscr{L}_i(\boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_j), \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_i + \boldsymbol{e}_j)) - \mathscr{L}_i(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_i)) \\ &\qquad - \mathscr{L}_j(\boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_i), \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_j + \boldsymbol{e}_i)) + \mathscr{L}_j(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_j)) = 0\ . \end{aligned} \tag{3.12}$$

Equation (3.12) represents the closure relation of the Lagrangian 1-form for the RS system and it can be explicitly shown holding on the level of the equations of motion, and as well as constraints.

Choosing a discrete curve Γ consisting of connected elements Γ_i, we can define an action on the curve by summing up the contributions $\mathscr{L}_i$ from each of the oriented links Γ_i in the curve, to get

$$S(\boldsymbol{x}(\mathbf{n}); \Gamma) = \sum_{\boldsymbol{n} \in \Gamma} \mathscr{L}_i(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_i)). \tag{3.13}$$

The closure relation (3.12) is actually equivalent to the invariance of the action under local deformations of the curve. To see this, suppose we have an action S evaluated on a curve Γ, and we deform this (keeping end points fixed) to get a curve Γ' on which an action S' is evaluated, such as in Figure 1.

Then S' is related to S by the following:

$$\begin{aligned} S' &= S - \mathscr{L}_i(\boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_j), \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_i + \boldsymbol{e}_j)) + \mathscr{L}_i(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_i)) \\ &\quad + \mathscr{L}_j(\boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_i), \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_j + \boldsymbol{e}_i)) - \mathscr{L}_j(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_j)). \end{aligned} \tag{3.14}$$

Equation (3.14) shows that the independence of the action under such a deformation is locally equivalent to the closure relation. The invariance of the action under the local deformation is a crucial aspect of the underlying variational principle. In Appendix C, we demonstrate how to derive the discrete Euler-Lagrange equation the variational for some specific discrete curves.

4 The semi-continuous limit: The skew limit

In this Section, we study a continuum analogue of a previous construction in Section 2 by considering a particular semi-continuous limit. Since the exact solution (2.25a) contains two discrete variables n and m, we could perform a continuum limit on one of these variables separately, while leaving the other discrete variable intact, and thus obtain a semi-continuous equation with one remaining discrete and two continuous independent variables. Alternatively, we can first perform a change of independent variables on the lattice and subsequently perform the limit on one of the new variables. The advantage of the latter approach over the former is that it often leads in a more direct way to a hierarchy of higher order flows. Adopting the latter approach in this section, we use a new discrete variable $\mathsf{N} := n + m$, and perform the transformation on the dependent variables by setting $x(n,m) \mapsto \mathsf{x}(\mathsf{N}, m) =: \mathsf{x}$, which leads to the following expressions for the shifted variables:

$$\begin{aligned} x = x(n+1, m) &\mapsto \mathsf{x}(\mathsf{N}+1, m) =: \widetilde{\mathsf{x}}\ , \\ \widehat{x} = x(n, m+1) &\mapsto \mathsf{x}(\mathsf{N}+1, m+1) =: \widehat{\widetilde{\mathsf{x}}}\ , \\ \widetilde{x} = x(n+1, m+1) &\mapsto \mathsf{x}(\mathsf{N}+2, m+1) =: \widehat{\widetilde{\widetilde{\mathsf{x}}}}\ . \end{aligned}$$

Rearranging the terms in (2.25a), we have

$$\begin{aligned}\boldsymbol{Y}(\mathsf{N},m) &= (p\boldsymbol{I}+\boldsymbol{\Lambda})^{-\mathsf{N}}\left(\frac{q\boldsymbol{I}+\boldsymbol{\Lambda}}{p\boldsymbol{I}+\boldsymbol{\Lambda}}\right)^{-m}\Bigg[\boldsymbol{Y}(0,0)-\frac{\mathsf{N}p\lambda}{p\boldsymbol{I}+\boldsymbol{\Lambda}} \\ &\quad +m\lambda\left(\frac{p}{p\boldsymbol{I}+\boldsymbol{\Lambda}}-\frac{q}{q\boldsymbol{I}+\boldsymbol{\Lambda}}\right)\Bigg](p\boldsymbol{I}+\boldsymbol{\Lambda})^{\mathsf{N}}\left(\frac{q\boldsymbol{I}+\boldsymbol{\Lambda}}{p\boldsymbol{I}+\boldsymbol{\Lambda}}\right)^{m} . \end{aligned} \tag{4.1}$$

We perform the limit $n\to-\infty$, $m\to\infty$, $\varepsilon\to 0$ while keeping N fixed and setting $\varepsilon=p-q$, such that $\varepsilon m=\tau$ remains finite. Focusing on the penultimate factor in (4.1) we have that

$$\lim_{\substack{m\to\infty\\ \varepsilon\to 0\\ \varepsilon m\to\tau}}\left(1-\frac{\varepsilon}{p\boldsymbol{I}+\boldsymbol{\Lambda}}\right)^{m}=\lim_{m\to\infty}\left(1-\frac{\tau}{m(p\boldsymbol{I}+\boldsymbol{\Lambda})}\right)^{m}=e^{-\frac{\tau}{p\boldsymbol{I}+\boldsymbol{\Lambda}}}, \tag{4.2}$$

so that the exact solution takes the form

$$\boldsymbol{Y}(\mathsf{N},\tau)=(p\boldsymbol{I}+\boldsymbol{\Lambda})^{-\mathsf{N}}e^{\frac{\tau}{p\boldsymbol{I}+\boldsymbol{\Lambda}}}\left[\boldsymbol{Y}(0,0)-\frac{\mathsf{N}p\lambda}{p\boldsymbol{I}+\boldsymbol{\Lambda}}+\frac{\tau\lambda\boldsymbol{\Lambda}}{(p\boldsymbol{I}+\boldsymbol{\Lambda})^2}\right](p\boldsymbol{I}+\boldsymbol{\Lambda})^{\mathsf{N}}e^{-\frac{\tau}{p\boldsymbol{I}+\boldsymbol{\Lambda}}} . \tag{4.3}$$

This equation represents the full solution after taking the skew limit. The position of the particles $\mathrm{x}_i(\mathsf{N},\tau)$ can be determined by computing the eigenvalues of (4.3).

4.1 The skew limit on equations of motion and constraints

We rewrite the equations of motion (3.2) in terms of the variables (N,m)

$$\begin{aligned}\sum_{\substack{j=1\\ j\neq i}}^{N}\left(\ln(\mathrm{x}_i-\mathrm{x}_j+\lambda)-\ln(\mathrm{x}_i-\mathrm{x}_j-\lambda)\right) &= \sum_{j=1}^{N}\Big(\ln(\mathrm{x}_i-\widetilde{\widetilde{\mathrm{x}}}_j)-\ln(\mathrm{x}_i-\underset{\sim}{\mathrm{x}}_j+\lambda) \\ &\quad +\ln(\mathrm{x}_i-\underset{\sim}{\mathrm{x}}_j)-\ln(\mathrm{x}_i-\widetilde{\widetilde{\mathrm{x}}}_j-\lambda)\Big) , \end{aligned} \tag{4.4}$$

Using a Taylor expansion

$$\widehat{\mathrm{x}} = \mathrm{x}(\tau+\varepsilon)=\mathrm{x}(\tau)+\varepsilon\frac{\partial \mathrm{x}}{\partial\tau}+\frac{\varepsilon^2}{2}\frac{\partial^2\mathrm{x}}{\partial\tau^2}+\ldots , \tag{4.5}$$

$$\underset{\frown}{\mathrm{x}} = \mathrm{x}(\tau-\varepsilon)=\mathrm{x}(\tau)-\varepsilon\frac{\partial \mathrm{x}}{\partial\tau}+\frac{\varepsilon^2}{2}\frac{\partial^2\mathrm{x}}{\partial\tau^2}+\ldots , \tag{4.6}$$

and collecting terms in order $\mathcal{O}(\varepsilon^0)$, we have the equations of motion for the RS system corresponding to the " N " variable

$$\begin{aligned}\sum_{\substack{j=1\\ j\neq i}}^{N}\left(\ln(\mathrm{x}_i-\mathrm{x}_j+\lambda)-\ln(\mathrm{x}_i-\mathrm{x}_j-\lambda)\right) &= \sum_{j=1}^{N}\Big(\ln(\mathrm{x}_i-\widetilde{\mathrm{x}}_j)-\ln(\mathrm{x}_i-\underset{\sim}{\mathrm{x}}_j+\lambda) \\ &\quad + \ln(\mathrm{x}_i-\underset{\sim}{\mathrm{x}}_j)-\ln(\mathrm{x}_i-\widetilde{\mathrm{x}}_j-\lambda)\Big) , \end{aligned} \tag{4.7}$$

and $\mathcal{O}(\varepsilon)$, we have

$$\sum_{j=1}^{N}\left[\frac{\partial\widetilde{\mathrm{x}}_j}{\partial\tau}\left(\frac{1}{\mathrm{x}_i-\widetilde{\mathrm{x}}_j-\lambda}-\frac{1}{\mathrm{x}_i-\widetilde{\mathrm{x}}_j}\right)+\frac{\partial\underset{\sim}{\mathrm{x}}_j}{\partial\tau}\left(\frac{1}{\mathrm{x}_i-\underset{\sim}{\mathrm{x}}_j+\lambda}-\frac{1}{\mathrm{x}_i-\underset{\sim}{\mathrm{x}}_j}\right)\right]=0 , \tag{4.8}$$

which are the equations of motion for the RS system corresponding to the τ variable.

Similarly, changing the variables to (N,τ) in (2.24a) and (2.24b) and collecting terms in order $\mathcal{O}(\varepsilon)$, we have

$$-\frac{1}{p} = \sum_{j=1}^{N}\frac{\partial\widetilde{\mathrm{x}}_j}{\partial\tau}\left(\frac{1}{\mathrm{x}_i-\widetilde{\mathrm{x}}_j-\lambda}-\frac{1}{\mathrm{x}_i-\widetilde{\mathrm{x}}_j}\right) , \tag{4.9}$$

$$\frac{1}{p} = \sum_{j=1}^{N}\frac{\partial\underset{\sim}{\mathrm{x}}_j}{\partial\tau}\left(\frac{1}{\mathrm{x}_i-\underset{\sim}{\mathrm{x}}_j+\lambda}-\frac{1}{\mathrm{x}_i-\underset{\sim}{\mathrm{x}}_j}\right) , \tag{4.10}$$

The summation of these two yields (4.8), while the difference gives

$$\sum_{j=1}^{N}\left[\frac{\partial\widetilde{\mathrm{x}}_j}{\partial\tau}\left(\frac{1}{\mathrm{x}_i-\widetilde{\mathrm{x}}_j-\lambda}-\frac{1}{\mathrm{x}_i-\widetilde{\mathrm{x}}_j}\right)-\frac{\partial\underset{\sim}{\mathrm{x}}_j}{\partial\tau}\left(\frac{1}{\mathrm{x}_i-\underset{\sim}{\mathrm{x}}_j+\lambda}-\frac{1}{\mathrm{x}_i-\underset{\sim}{\mathrm{x}}_j}\right)\right]=0\ , \tag{4.11}$$

which are the constraint equations after taking the skew limit.

4.2 The skew limit on action

In this Section, we proceed exactly the same steps what we did in [24]. First, we observe that eq. (4.7) can be once again be obtained by implementing the usual variational principle on the following action $S_{(\mathsf{N})}$ given by

$$\begin{aligned} S_{(\mathsf{N})}=\sum_{\mathsf{N}}\mathscr{L}_{(\mathsf{N})} &= \sum_{\mathsf{N}}\Bigg(\sum_{i,j=1}^{N}\left(f(\mathrm{x}_i-\widetilde{\mathrm{x}}_j)-f(\mathrm{x}_i-\widetilde{\mathrm{x}}_j-\lambda)\right)-\frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N}f(\mathrm{x}_i-\mathrm{x}_j+\lambda) \\ &\quad -\frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N}f(\widetilde{\mathrm{x}}_i-\widetilde{\mathrm{x}}_j+\lambda)-\ln\left|\frac{p}{\sqrt{\beta}}\right|\sum_{i=1}^{N}(\mathrm{x}_i-\widetilde{\mathrm{x}}_i)\Bigg)\ , \end{aligned} \tag{4.12}$$

where now the Lagrangian $\mathscr{L}_{(\mathsf{N})}$ involves variables $\widetilde{\mathsf{x}}_i$ shifted in the discrete variable N instead of the original variable n, and the corresponding discrete Euler-Lagrange equation reads:

$$\widetilde{\frac{\partial\mathscr{L}_{(\mathsf{N})}}{\partial\mathsf{x}_i}}+\left(\frac{\partial\mathscr{L}_{(\mathsf{N})}}{\partial\widetilde{\mathsf{x}}_i}\right)=0, \tag{4.13}$$

yielding (4.7).

Second, we observe that eq. (4.8) can be once again be obtained by implementing the usual variational principle on the following action $S_{(\tau)}$ given by

$$S_{(\tau)}=\int_{\tau_1}^{\tau_2}d\tau\mathscr{L}_{(\tau)}\left(\mathbf{x}(\mathsf{N}_0-1,\tau),\frac{\partial\mathbf{x}(\mathsf{N}_0,\tau)}{\partial\tau}\right)\ , \tag{4.14}$$

which is obtained by taking the skew limit together with anti-Taylor expansion of (3.4) and

$$\begin{aligned} \mathscr{L}_{(\tau)} &= \sum_{i,j=1}^{N}\left(\frac{\partial\widetilde{\mathrm{x}}_j}{\partial\tau}(\ln|\mathrm{x}_i-\widetilde{\mathrm{x}}_j-\lambda|-\ln|\mathrm{x}_i-\widetilde{\mathrm{x}}_j|)\right) \\ &\quad -\frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N}\left(\frac{\partial\widetilde{\mathrm{x}}_j}{\partial\tau}\left(\ln|\widetilde{\mathrm{x}}_i-\widetilde{\mathrm{x}}_j+\lambda|-\ln|\widetilde{\mathrm{x}}_i-\widetilde{\mathrm{x}}_j-\lambda|\right)+\frac{\partial\widetilde{x}_i}{\partial\tau}-\frac{\partial\widetilde{x}_j}{\partial\tau}\right) \\ &\quad +\sum_{i=1}^{N}\left(\frac{1}{p}(\mathrm{x}_i-\widetilde{\mathrm{x}}_i)+\frac{\partial\widetilde{\mathrm{x}}_i}{\partial\tau}\ln\left|p\sqrt{\beta}\right|\right)\ . \end{aligned} \tag{4.15}$$

The Euler Lagrange equations

$$\frac{\partial\mathscr{L}_{(\tau)}}{\partial\mathrm{x}_i}-\frac{d}{d\tau}\left(\frac{\partial\mathscr{L}_{(\tau)}}{\partial(d\mathrm{x}_i/d\tau)}\right)=0\ , \tag{4.16}$$

yield (4.8).

5 The full limit

In the previous Section, we took the continuum limit on the discrete variable m, leading to a system of differential-difference equations. The full continuum limit, performed on the

remaining discrete variable N, will lead to a coupled system of poles in the first instance, from which a hierarchy of ODEs can be retrieved, which is the RS hierarchy. How to perform this limit is inspired by the structure of the solutions of (4.3). Performing the following computation,

$$\begin{aligned}
\mathbf{Y}(\mathsf{N},\tau) &= \left(\boldsymbol{I}+\frac{\boldsymbol{\Lambda}}{p}\right)^{-\mathsf{N}} e^{\frac{\tau}{p}\left(\boldsymbol{I}+\frac{\boldsymbol{\Lambda}}{p}\right)^{-1}}\mathbf{Y}(0,0)e^{-\frac{\tau}{p}\left(\boldsymbol{I}+\frac{\boldsymbol{\Lambda}}{p}\right)^{-1}}\left(\boldsymbol{I}+\frac{\boldsymbol{\Lambda}}{p}\right)^{\mathsf{N}} \\
&\quad -\mathsf{N}\lambda\left(1+\frac{\boldsymbol{\Lambda}}{p}\right)^{-1}+\frac{\tau\lambda\boldsymbol{\Lambda}}{p^2}\left(1+\frac{\boldsymbol{\Lambda}}{p}\right)^{-2} \\
&= e^{-\mathsf{N}\ln\left(1+\frac{\boldsymbol{\Lambda}}{p}\right)+\frac{\tau}{p}\left(1+\frac{\boldsymbol{\Lambda}}{p}\right)^{-1}}\mathbf{Y}(0,0)e^{\mathsf{N}\ln\left(1+\frac{\boldsymbol{\Lambda}}{p}\right)-\frac{\tau}{p}\left(1+\frac{\boldsymbol{\Lambda}}{p}\right)^{-1}} \\
&\quad -\mathsf{N}\lambda\left(1+\frac{\boldsymbol{\Lambda}}{p}\right)^{-1}+\frac{\tau\lambda\boldsymbol{\Lambda}}{p^2}\left(1+\frac{\boldsymbol{\Lambda}}{p}\right)^{-2}\ .
\end{aligned} \tag{5.1}$$

We now introduce

$$t_1=\frac{\tau}{p^2}+\frac{\mathsf{N}}{p}\ ,\quad t_2=-\frac{2\tau}{p^3}-\frac{\mathsf{N}}{p^2}\ ,\quad t_3=\frac{3\tau}{p^4}+\frac{\mathsf{N}}{p^3}\ ,\qquad .....\ , \tag{5.2a}$$

and expand (5.1) with respect to variable p. We have

$$\begin{aligned}
\mathbf{Y}(t_1,t_2,t_3,...,\mathsf{N}) &= e^{-\boldsymbol{\Lambda}t_1+\boldsymbol{\Lambda}^2\frac{t_2}{2}-\boldsymbol{\Lambda}^3\frac{t_3}{3}+\cdots}\mathbf{Y}(0,0,...)e^{\boldsymbol{\Lambda}t_1-\boldsymbol{\Lambda}^2\frac{t_2}{2}+\boldsymbol{\Lambda}^3\frac{t_3}{3}+\cdots} \\
&\quad -\mathsf{N}\lambda+\boldsymbol{\Lambda}\lambda t_1+\boldsymbol{\Lambda}^2\lambda t_2+\boldsymbol{\Lambda}^3\lambda t_3+...\ ,
\end{aligned} \tag{5.3}$$

which is a function of time variables $(t_1,t_2,t_3,...,\mathsf{N})$. The positions of the particles $X_i(t_1,t_2,t_3,...,\mathsf{N})$ can be computed by looking for the eigenvalues of (5.3). The explicit expression of the solution for the RS can be obtained from the secular problem for the matrix

$$\boldsymbol{X}(0,0)-\xi+\boldsymbol{L}(0,0)\lambda t_1+\boldsymbol{L}^2(0,0)\lambda t_2+\boldsymbol{L}^3(0,0)\lambda t_3\ , \tag{5.4}$$

where $\xi=\mathsf{N}\lambda$ and $\boldsymbol{X}(0,0)=\boldsymbol{U}^{-1}(0,0)\boldsymbol{Y}(0,0)\boldsymbol{U}(0,0)$ and $\boldsymbol{L}(0,0)=\boldsymbol{U}^{-1}(0,0)\boldsymbol{\Lambda}\boldsymbol{U}(0,0)$. The solution (5.3) involves N-time flows for the RS system. The next solutions in the hierarchy can be generated by pushing further on with the expansion.

5.1 The full limit on the equations of motion

We now would like to see what would result from taking the limit on the equations of motion (4.8). First, we introduce

$$\begin{aligned}
\dot{\mathsf{x}}_i &= \frac{\partial \mathsf{x}_i}{\partial\tau}=\frac{\partial X_i}{\partial t_1}\frac{\partial t_1}{\partial\tau}+\frac{\partial X_i}{\partial t_2}\frac{\partial t_2}{\partial\tau}+\frac{\partial X_i}{\partial t_3}\frac{\partial t_3}{\partial\tau}+... \\
&= \frac{1}{p^2}\frac{\partial X_i}{\partial t_1}-\frac{2}{p^3}\frac{\partial X_i}{\partial t_2}+\frac{3}{p^4}\frac{\partial X_i}{\partial t_3}+...\ ,
\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}
\widetilde{\mathsf{x}}_i &= \mathsf{x}_i-\lambda+\frac{1}{p}\frac{\partial X_i}{\partial t_1}+\frac{1}{p^2}\left(\frac{1}{2}\frac{\partial^2 X_i}{\partial t_1^2}-\frac{\partial X_i}{\partial t_2}\right)+\frac{1}{p^3}\left(\frac{1}{6}\frac{\partial X_i}{\partial t_3}-\frac{\partial^2 X_i}{\partial t_1\partial t_2}+\frac{\partial X_i}{\partial t_3}\right) \\
&\quad +\frac{1}{p^4}\left(\frac{1}{24}\frac{\partial^4 X_i}{\partial t_4^4}-\frac{1}{2}\frac{\partial^3 X_i}{\partial t_1^2\partial t_2}+\frac{1}{2}\frac{\partial^2 X_i}{\partial t_2^2}+\frac{\partial^2 X_i}{\partial t_1\partial t_3}\right)+\mathcal{O}(1/p^5)\ ,
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
\underset{\sim}{\mathsf{x}}_i &= \mathsf{x}_i+\lambda-\frac{1}{p}\frac{\partial X_i}{\partial t_1}+\frac{1}{p^2}\left(\frac{1}{2}\frac{\partial^2 X_i}{\partial t_1^2}+\frac{\partial X_i}{\partial t_2}\right)+\frac{1}{p^3}\left(-\frac{1}{6}\frac{\partial X_i}{\partial t_3}-\frac{\partial^2 X_i}{\partial t_1\partial t_2}-\frac{\partial X_i}{\partial t_3}\right) \\
&\quad +\frac{1}{p^4}\left(\frac{1}{24}\frac{\partial^4 X_i}{\partial t_4^4}+\frac{1}{2}\frac{\partial^3 X_i}{\partial t_1^2\partial t_2}+\frac{1}{2}\frac{\partial^2 X_i}{\partial t_2^2}+\frac{\partial^2 X_i}{\partial t_1\partial t_3}\right)+\mathcal{O}(1/p^5)\ .
\end{aligned} \tag{5.7}$$

Then we expand (4.8) with respect to the variable p together with (5.2a). We find that

♦The leading term of order $\mathcal{O}(1/p^3)$ gives us

$$\frac{\partial^2 X_i}{\partial t_1^2}\Big/\frac{\partial X_i}{\partial t_1}+\sum_{j=1}^{N}\frac{\partial X_j}{\partial t_1}\left(\frac{1}{X_i-X_j+\lambda}+\frac{1}{X_i-X_j-\lambda}-\frac{2}{X_i-X_j}\right)=0\;, \tag{5.8}$$

which is the equations of the motion for the continuous RS system.

♦The term of order $\mathcal{O}(1/p^4)$ gives us

$$\begin{aligned}&2\frac{\partial^2 X_i}{\partial t_1 t_2}\Big/\frac{\partial X_i}{\partial t_1}-\frac{\partial^2 X_i}{\partial t_1^2}\frac{\partial X_i}{\partial t_2}\Big/\left(\frac{\partial X_i}{\partial t_2}\right)^2-\frac{1}{\lambda}\frac{\partial^2 X_i}{\partial t_1^2}\\ &+\sum_{j=1}^{N}\left[\frac{\partial X_j}{\partial t_2}\left(\frac{1}{X_i-X_j+\lambda}+\frac{1}{X_i-X_j-\lambda}-\frac{2}{X_i-X_j}\right)\right.\\ &+\frac{1}{2}\frac{\partial^2 X_j}{\partial t_1^2}\left(\frac{1}{X_i-X_j-\lambda}-\frac{1}{X_i-X_j+\lambda}\right)\\ &\left.+\frac{1}{2}\left(\frac{\partial X_j}{\partial t_2}\right)^2\left(\frac{1}{(X_i-X_j-\lambda)^2}-\frac{1}{(X_i-X_j+\lambda)^2}\right)\right]=0\;. \end{aligned}\tag{5.9}$$

This equation represents the next equation of motion beyond the usual continuous RS in the hierarchy. We will stop at this equation, but we can actually get the higher terms of the equation in which the variable t_3 and higher order time-flows must be taken into account.

The full limit of (4.11) in the order $\mathcal{O}(1/p^2)$ gives

$$\frac{2}{\lambda}\frac{\partial X_i}{\partial t_1}-2\frac{\partial X_i}{\partial t_2}\Big/\frac{\partial X_i}{\partial t_1}+\sum_{j=1}^{N}\frac{\partial X_j}{\partial t_1}\left(\frac{1}{X_i-X_j+\lambda}-\frac{1}{X_i-X_j-\lambda}\right)=0\;, \tag{5.10}$$

which is the constraint equations for the full limit.

Using (5.10), we can simplify (5.9) into

$$\begin{aligned}&\frac{\partial^2 X_i}{\partial t_1 t_2}\Big/\frac{\partial X_i}{\partial t_1}+\sum_{j=1}^{N}\left[\frac{\partial X_j}{\partial t_2}\left(\frac{1}{X_i-X_j+\lambda}+\frac{1}{X_i-X_j-\lambda}-\frac{2}{X_i-X_j}\right)\right.\\ &\left.-\frac{1}{2}\frac{\partial X_i}{\partial t_1}\frac{\partial X_j}{\partial t_1}\left(\frac{1}{(X_i-X_j-\lambda)^2}-\frac{1}{(X_i-X_j+\lambda)^2}\right)\right]=0\;. \end{aligned}\tag{5.11}$$

Note that (5.11) can be obtained directly from the full limit in order $\mathcal{O}(1/p^3)$ from the combination of the relations

$$-\frac{1}{p}=\sum_{j=1}^{N}\frac{\partial \mathrm{x}_j}{\partial\tau}\left(\frac{1}{\underset{\sim}{\mathrm{x}}_i-\mathrm{x}_j-\lambda}-\frac{1}{\underset{\sim}{\mathrm{x}}_i-\mathrm{x}_j}\right)\;, \tag{5.12}$$

$$\frac{1}{p}=\sum_{j=1}^{N}\frac{\partial \mathrm{x}_j}{\partial\tau}\left(\frac{1}{\widetilde{\mathrm{x}}_i-\mathrm{x}_j-\lambda}-\frac{1}{\widetilde{\mathrm{x}}_i-\mathrm{x}_j}\right)\;, \tag{5.13}$$

which are the backward shift and forward shift of (4.9) and (4.10), respectively.

5.2 The full limit on the action

We will follow the steps in [24] in order to obtain the full limit on the action. We now take the action to be of the form

$$S[\boldsymbol{x}(\mathsf{N},\tau);\Gamma]=\int_{\tau_1}^{\tau_2}d\tau\mathscr{L}_{(\tau)}(\boldsymbol{x}(\mathsf{N},\tau),\dot{\boldsymbol{x}}(\mathsf{N},\tau))+\sum_{\mathsf{N}}\mathscr{L}_{(\mathsf{N})}(\boldsymbol{x}(\mathsf{N},\tau),\boldsymbol{x}(\mathsf{N}+1,\tau))\;, \tag{5.14}$$

where the first term belongs to the vertical part and the second term belongs to the horizontal part of the curve Γ.

Using anti-Taylor expansion, the action now becomes

$$S[\boldsymbol{x}(\mathsf{N},\tau);\Gamma]=\int_{\tau_1}^{\tau_2}d\tau\mathscr{L}_{(\tau)}(\boldsymbol{x}(\mathsf{N},\tau),\dot{\boldsymbol{x}}(\mathsf{N},\tau))+\int_{\mathsf{N}_1}^{\mathsf{N}_2}d\mathsf{N}\mathscr{L}_{(\mathsf{N})}(\boldsymbol{x}(\mathsf{N},\tau),\boldsymbol{x}(\mathsf{N}+1,\tau))\ , \quad (5.15)$$

where we do not need to take into account the boundary terms coming from the expansion, because they are constant at the end points and do not contribute to the variational process.

We now perform a change of variables $(\tau,\mathsf{N})\mapsto(t_1,t_2)$ by using (5.2a)

$$\begin{aligned} d\tau &= -p^3dt_2-p^2dt_1\ , && (5.16a)\\ d\mathsf{N} &= p^2dt_2+2pdt_1\ , && (5.16b)\end{aligned}$$

and also expand the Lagrangians with respect to variable p. We obtain

$$\begin{aligned} S[\boldsymbol{X}(t_1,t_2);\Gamma] &= \int_{t_1(1)}^{t_1(2)}dt_1\mathscr{L}_{(t_1)}\left(\boldsymbol{X}(t_1,t_2),\frac{\partial\boldsymbol{X}(t_1,t_2)}{\partial t_1}\right)\\ &+ \int_{t_2(1)}^{t_2(2)}dt_3\mathscr{L}_{(t_2)}\left(\boldsymbol{X}(t_1,t_2),\frac{\partial\boldsymbol{X}(t_1,t_2)}{\partial t_1},\frac{\partial\boldsymbol{X}(t_1,t_2)}{\partial t_2}\right)\ , \quad (5.17)\end{aligned}$$

where $\mathscr{L}_{(t_1)}$ and $\mathscr{L}_{(t_2)}$ are given by

$$\mathscr{L}_{(t_1)}=\sum_{i=1}^{N}\frac{\partial X_i}{\partial t_1}\ln\left|\frac{\partial X_i}{\partial t_1}\right|-\sum_{i\neq j}^{N}\frac{\partial X_j}{\partial t_1}\left(\ln|X_i-X_j-\lambda|-\ln|X_i-X_j|\right)\ , \quad (5.18)$$

of which the Euler-Lagrange equation

$$\frac{\partial\mathscr{L}_{(t_1)}}{\partial X_i}-\frac{\partial}{\partial t_1}\left(\frac{\partial\mathscr{L}_{(t_1)}}{\partial(\frac{\partial X_i}{\partial t_1})}\right)=0\ , \quad (5.19)$$

gives exactly eq. (5.8) and

$$\begin{aligned}\mathscr{L}_{(t_2)} &= \sum_{i=1}^{N}\left(\frac{\partial X_i}{\partial t_2}\ln\left|\frac{\partial X_i}{\partial t_1}\right|-\frac{1}{2\lambda}\left(\frac{\partial X_i}{\partial t_1}\right)^2+3\frac{\partial X_i}{\partial t_2}\right)\\ &-\sum_{i\neq j}^{N}\left[\frac{\partial X_j}{\partial t_2}\left(\ln|X_i-X_j-\lambda|-\ln|X_i-X_j|\right)\right.\\ &\left.+\frac{1}{2}\frac{\partial X_i}{\partial t_1}\frac{\partial X_j}{\partial t_1}\frac{1}{X_i-X_j+\lambda}\right] \quad (5.20)\end{aligned}$$

We see that the Lagrangian $\mathscr{L}_{(t_2)}$ contains derivatives with respect to two time flows t_1 and t_2. We observe that the equations of motion (5.11) require the Euler-Lagrange equation in the form

$$\frac{\partial\mathscr{L}_{(t_2)}}{\partial X_i}-\frac{\partial}{\partial t_2}\left(\frac{\partial\mathscr{L}_{(t_2)}}{\partial(\frac{\partial X_i}{\partial t_2})}\right)=0\ . \quad (5.21)$$

Furthermore, we find that

$$\frac{\partial\mathscr{L}_{(t_2)}}{\partial(\frac{\partial X_i}{\partial t_1})}=\frac{2}{\lambda}\frac{\partial X_i}{\partial t_1}-2\frac{\partial X_i}{\partial t_2}\Bigg/\frac{\partial X_i}{\partial t_1}+\sum_{j=1}^{N}\frac{\partial X_j}{\partial t_1}\left(\frac{1}{X_i-X_j+\lambda}-\frac{1}{X_i-X_j-\lambda}\right)=0\ , \quad (5.22)$$

which is identical to term with the full limit in order $\mathcal{O}(1/p^2)$ of the constraints (5.10).

Here we obtained the hierarchy of Lagrangians for the RS-model through the full continuum limit. Obviously, higher Lagrangians in the family can be generated by pushing further on with the expansion.

5.3 The full limit on the closure relation

We take the full limit on the discrete closure relation 3.7 leading to

Theorem 5.1. *We find that the continuous version of the closure relation between t_1 and t_2*

$$\frac{\partial \mathscr{L}_{(t_2)}}{\partial t_1} = \frac{\partial \mathscr{L}_{(t_1)}}{\partial t_2} \, , \tag{5.23}$$

which holds on the equations of motion and constraint.

Proof. : We find that

$$\begin{aligned}
\frac{\partial \mathscr{L}_{(t_1)}}{\partial t_2} \;\; = \;\; & \sum_{i=1}^{N} \left(\frac{\partial^2 X_i}{\partial t_1 \partial t_2} \ln \left| \frac{\partial X_i}{\partial t_1} \right| + \frac{\partial^2 X_i}{\partial t_1 \partial t_2} \right) \\
& - \sum_{i \neq j}^{N} \left(\frac{\partial^2 X_j}{\partial t_2 \partial t_1} \left[\ln |X_i - X_j - \lambda| - \ln |X_i - X_j| \right] \right. \\
& + \frac{\partial X_j}{\partial t_1} \frac{\partial X_i}{\partial t_2} \left[\frac{1}{X_i - X_j - \lambda} - \frac{1}{X_i - X_j} \right] \\
& \left. + \frac{\partial X_j}{\partial t_1} \frac{\partial X_j}{\partial t_2} \left[\frac{1}{X_i - X_j - \lambda} - \frac{1}{X_i - X_j} \right] \right) \, ,
\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}
\frac{\partial \mathscr{L}_{(t_2)}}{\partial t_1} \;\; = \;\; & \sum_{i=1}^{N} \left(\frac{\partial^2 X_i}{\partial t_1 \partial t_2} \ln \left| \frac{\partial X_i}{\partial t_1} \right| + \frac{\partial X_i}{\partial t_2} \frac{\partial^2 X_i}{\partial t_1^2} \Big/ \frac{\partial X_i}{\partial t_1} - \frac{1}{\lambda} \frac{\partial X_i}{\partial t_1} \frac{\partial^2 X_i}{\partial t_1^2} + 3 \frac{\partial^2 X_i}{\partial t_1 \partial t_2} \right) \\
& - \sum_{i \neq j}^{N} \left(\frac{\partial^2 X_j}{\partial t_2 \partial t_1} \left[\ln |X_i - X_j - \lambda| - \ln |X_i - X_j| \right] \right. \\
& + \frac{\partial X_i}{\partial t_1} \frac{\partial X_j}{\partial t_2} \left[\frac{1}{X_i - X_j - \lambda} - \frac{1}{X_i - X_j} \right] \\
& + \frac{\partial X_j}{\partial t_1} \frac{\partial X_j}{\partial t_2} \left[\frac{1}{X_i - X_j - \lambda} - \frac{1}{X_i - X_j} \right] \\
& + \frac{1}{2} \frac{\partial^2 X_j}{\partial t_1^2} \frac{\partial X_i}{\partial t_1} \left[\frac{1}{X_i - X_j + \lambda} - \frac{1}{X_i - X_j - \lambda} \right] \\
& \left. - \frac{1}{2} \left(\frac{\partial X_j}{\partial t_1} \right)^2 \frac{\partial X_i}{\partial t_1} \left[\frac{1}{(X_i - X_j - \lambda)^2} - \frac{1}{(X_i - X_j + \lambda)^2} \right] \right) \, .
\end{aligned} \tag{5.25}$$

We find that $\frac{\partial \mathscr{L}_{(t_1)}}{\partial t_2} = \frac{\partial \mathscr{L}_{(t_2)}}{\partial t_1}$ gives

$$\begin{aligned}
& - \sum_{i=1}^{N} \frac{\partial^2 X_i}{\partial t_1 \partial t_2} + \frac{\partial X_j}{\partial t_1} \frac{\partial X_i}{\partial t_2} \left[\frac{1}{X_i - X_j - \lambda} - \frac{1}{X_i - X_j} \right] = \sum_{i=1}^{N} \left(\frac{\partial X_i}{\partial t_2} \frac{\partial^2 X_i}{\partial t_1^2} \Big/ \frac{\partial X_i}{\partial t_1} \right. \\
& \left. - \frac{1}{\lambda} \frac{\partial X_i}{\partial t_1} \frac{\partial^2 X_i}{\partial t_1^2} + \frac{\partial^2 X_i}{\partial t_1 \partial t_2} \right) + \sum_{i \neq j}^{N} \left(- \frac{\partial X_i}{\partial t_1} \frac{\partial X_j}{\partial t_2} \left[\frac{1}{X_i - X_j - \lambda} - \frac{1}{X_i - X_j} \right] \right. \\
& + \frac{1}{2} \frac{\partial^2 X_j}{\partial t_1^2} \frac{\partial X_i}{\partial t_1} \left[\frac{1}{X_i - X_j + \lambda} - \frac{1}{X_i - X_j - \lambda} \right] \\
& \left. - \frac{1}{2} \left(\frac{\partial X_j}{\partial t_1} \right)^2 \frac{\partial X_i}{\partial t_1} \left[\frac{1}{(X_i - X_j - \lambda)^2} - \frac{1}{(X_i - X_j + \lambda)^2} \right] \right) \, .
\end{aligned} \tag{5.26}$$

Dividing (5.26) by $\frac{\partial X_i}{\partial t_1}$ we find that

$$\frac{\partial \mathscr{L}_{(t_1)}}{\partial t_2} - \frac{\partial \mathscr{L}_{(t_2)}}{\partial t_1} = \sum_{i=1}^{N} \frac{\partial X_i}{\partial t_1} \left(2\frac{\partial^2 X_i}{\partial t_1 t_2} \Big/ \frac{\partial X_i}{\partial t_1} - \frac{\partial^2 X_i}{\partial t_1^2}\frac{\partial X_i}{\partial t_2} \Big/ \left(\frac{\partial X_i}{\partial t_2}\right)^2 - \frac{1}{\lambda}\frac{\partial^2 X_i}{\partial t_1^2} \right.$$
$$-\sum_{j=1}^{N} \left[\frac{\partial X_j}{\partial t_2} \left(\frac{1}{X_i - X_j + \lambda} + \frac{1}{X_i - X_j - \lambda} - \frac{2}{X_i - X_j} \right) \right.$$
$$+\frac{1}{2}\frac{\partial^2 X_j}{\partial t_1^2} \left(\frac{1}{X_i - X_j + \lambda} + \frac{1}{X_i - X_j - \lambda} \right)$$
$$\left.\left. - \frac{1}{2}\left(\frac{\partial X_i}{\partial t_2}\right)^2 \left(\frac{1}{(X_i - X_j - \lambda)^2} - \frac{1}{(X_i - X_j + \lambda)^2} \right) \right] \right) . \tag{5.27}$$

Using (5.9), (5.27) becomes

$$\frac{\partial \mathscr{L}_{(t_1)}}{\partial t_2} - \frac{\partial \mathscr{L}_{(t_2)}}{\partial t_1} = \sum_{i=1}^{N} \frac{\partial X_i}{\partial t_1} \left(2\frac{\partial^2 X_i}{\partial t_1^2}\frac{\partial X_i}{\partial t_2} \Big/ \left(\frac{\partial X_i}{\partial t_2}\right)^2 \right.$$
$$\left. -2\sum_{j=1}^{N} \frac{\partial X_j}{\partial t_2} \left(\frac{1}{X_i - X_j + \lambda} + \frac{1}{X_i - X_j - \lambda} - \frac{2}{X_i - X_j} \right) \right) . \tag{5.28}$$

Inserting (5.8), we have now

$$\frac{\partial \mathscr{L}_{(t_1)}}{\partial t_2} - \frac{\partial \mathscr{L}_{(t_2)}}{\partial t_1}$$
$$= -2\sum_{i,j=1}^{N} \left(\frac{\partial X_j}{\partial t_1}\frac{\partial X_i}{\partial t_2} + \frac{\partial X_i}{\partial t_1}\frac{\partial X_j}{\partial t_2} \right) \left(\frac{1}{X_i - X_j + \lambda} + \frac{1}{X_i - X_j - \lambda} - \frac{2}{X_i - X_j} \right) \tag{5.29}$$

The first term of (5.29) is the antisymmetric function, hence vanishes. We now have

$$\frac{\partial \mathscr{L}_{(t_1)}}{\partial t_2} - \frac{\partial \mathscr{L}_{(t_2)}}{\partial t_1} = 0 . \tag{5.30}$$

□

6 The connection to the lattice KP systems

In contrast to the CM case [24], where we started with a semi-discrete KP equation, and applied a pole-reduction to it to a yield a compatible CM system, here we start from RS system and reconnect it to the fully discrete lattice KP systems. In [15], the connection between the RS system and the KP system was established for the trigonometric case, but here we will focus on the (simpler) rational case as it clarifies the situation more clearly,

We will develop now a scheme along the lines of the papers [26, 8, 9, 25]. Starting from the "solution matrix" $\boldsymbol{Y}(n,m)$ of (2.25a), we will introduce the relevant τ-function as its characteristic polynomial:

$$\tau(\xi) = \det(\xi \boldsymbol{I} - \boldsymbol{Y}) , \tag{6.1}$$

where $\boldsymbol{Y} = \boldsymbol{Y}(n,m,h)$ is a function of three discrete variables

$$\widetilde{\boldsymbol{Y}} = (p\boldsymbol{I} + \boldsymbol{\Lambda})^{-1}\boldsymbol{Y}(p\boldsymbol{I} + \boldsymbol{\Lambda}) - \frac{p\lambda}{p\boldsymbol{I} + \boldsymbol{\Lambda}} , \tag{6.2a}$$

$$\widehat{\boldsymbol{Y}} = (q\boldsymbol{I} + \boldsymbol{\Lambda})^{-1}\boldsymbol{Y}(q\boldsymbol{I} + \boldsymbol{\Lambda}) - \frac{q\lambda}{q\boldsymbol{I} + \boldsymbol{\Lambda}} , \tag{6.2b}$$

$$\overline{\boldsymbol{Y}} = (r\boldsymbol{I} + \boldsymbol{\Lambda})^{-1}\boldsymbol{Y}(r\boldsymbol{I} + \boldsymbol{\Lambda}) - \frac{r\lambda}{r\boldsymbol{I} + \boldsymbol{\Lambda}} , \tag{6.2c}$$

with r is a lattice parameter corresponding to the " $\bar{}$ " and we show that $\tau(\xi)$ plays the role of the tau-function of a discrete soliton system.

To derive the equations directly from the resolvent of the matrix $\boldsymbol{Y}$, we proceed as follows. First, we perform the simple computation

$$\begin{aligned}
\widetilde{\tau}(\xi) &= \det(\xi + \lambda - \boldsymbol{Y} - \widetilde{\boldsymbol{r}}\boldsymbol{s}^T) , \\
&= \det((\xi + \lambda - \boldsymbol{Y})(1 - \widetilde{\boldsymbol{r}}\boldsymbol{s}^T(\xi + \lambda - \boldsymbol{Y})^{-1})) , \\
&= \tau(\xi + \lambda)(1 - \boldsymbol{s}^T(\xi + \lambda - \boldsymbol{Y})^{-1}\widetilde{\boldsymbol{r}}) ,
\end{aligned}$$

then we have

$$\frac{\widetilde{\tau}(\xi)}{\tau(\xi + \lambda)} = 1 - \boldsymbol{s}^T(\xi + \lambda - \boldsymbol{Y})^{-1}(p + \boldsymbol{\Lambda})^{-1}\boldsymbol{r} = \mathbf{v}_p(\xi + \lambda) . \tag{6.3}$$

The reverse fraction of Eq. (6.3) can be computed by processing the same computation

$$\begin{aligned}
\tau(\xi) &= \det(\xi - \lambda - \widetilde{\boldsymbol{Y}} + \widetilde{\boldsymbol{r}}\boldsymbol{s}^T) , \\
&= \det((\xi - \lambda - \widetilde{\boldsymbol{Y}})(1 + \widetilde{\boldsymbol{r}}\boldsymbol{s}^T(\xi - \lambda - \widetilde{\boldsymbol{Y}})^{-1})) , \\
&= \widetilde{\tau}(\xi - \lambda)(1 + \widetilde{\boldsymbol{s}}^T(\xi - \lambda - \widetilde{\boldsymbol{Y}})^{-1}\widetilde{\boldsymbol{r}}) ,
\end{aligned}$$

then we have

$$\frac{\tau(\xi)}{\widetilde{\tau}(\xi - \lambda)} = 1 + \widetilde{\boldsymbol{s}}^T(p + \boldsymbol{\Lambda})^{-1}(\xi - \lambda - \widetilde{\boldsymbol{Y}})^{-1}\widetilde{\boldsymbol{r}} = \widetilde{\mathbf{w}}_p(\xi - \lambda) . \tag{6.4}$$

From (6.3) and (6.4), we have the relation

$$\frac{\tau(\xi)}{\widetilde{\tau}(\xi - \lambda)} = \widetilde{\mathbf{w}}_p(\xi - \lambda) = \frac{1}{\mathbf{v}_p(\xi)} . \tag{6.5}$$

The same type of the relation for the other discrete direction can be obtained in the forms

$$\frac{\tau(\xi)}{\widehat{\tau}(\xi - \mu)} = \widehat{\mathbf{w}}_q(\xi - \mu) = \frac{1}{\mathbf{v}_q(\xi)} , \tag{6.6a}$$

$$\frac{\tau(\xi)}{\overline{\tau}(\xi - \eta)} = \overline{\mathbf{w}}_r(\xi - \eta) = \frac{1}{\mathbf{v}_r(\xi)} . \tag{6.6b}$$

In order to derive discrete KP equations for $\tau(\xi)$, $\mathbf{w}$ and $\mathbf{v}$, we introduce the N-component vectors

$$\boldsymbol{u}_a(\xi) = (\xi - \boldsymbol{Y})^{-1}(a + \boldsymbol{\Lambda})^{-1}\boldsymbol{r} , \tag{6.7a}$$

$${}^{t}\boldsymbol{u}_b(\xi) = \boldsymbol{s}^T(b + \boldsymbol{\Lambda})^{-1}(\xi - \boldsymbol{Y})^{-1} , \tag{6.7b}$$

as well as the scalar variables

$$S_{ab}(\xi) = \boldsymbol{s}^T(b + \boldsymbol{\Lambda})^{-1}(\xi - \boldsymbol{Y})^{-1}(a + \boldsymbol{\Lambda})^{-1}\boldsymbol{r} . \tag{6.8}$$

We now consider (6.7a) which can be written in the form

$$\begin{aligned}
(\xi - \boldsymbol{Y})\boldsymbol{u}_a(\xi) &= (a + \boldsymbol{\Lambda})^{-1}(p + \boldsymbol{\Lambda})\widetilde{\boldsymbol{r}} , \\
(\xi - \lambda - \widetilde{\boldsymbol{Y}} + \widetilde{\boldsymbol{r}}\boldsymbol{s}^T)\boldsymbol{u}_a(\xi) &= (p - a)(a + \boldsymbol{\Lambda})^{-1}\widetilde{\boldsymbol{r}} + \widetilde{\boldsymbol{r}} , \\
(\xi - \lambda - \widetilde{\boldsymbol{Y}})\boldsymbol{u}_a(\xi) &= (p - a)(a + \boldsymbol{\Lambda})^{-1}\widetilde{\boldsymbol{r}} + \widetilde{\boldsymbol{r}}(1 - \boldsymbol{s}^T\boldsymbol{u}_a(\xi)) , \\
\boldsymbol{u}_a(\xi) &= (p - a)\widetilde{\boldsymbol{u}}_a(\xi - \lambda) + \mathbf{v}_a(\xi)\widetilde{\boldsymbol{u}}_0(\xi - \lambda) , \qquad (6.9)
\end{aligned}$$

with $\boldsymbol{u}_0(\xi) = (\xi - \boldsymbol{Y})^{-1}\boldsymbol{r}$.

The same process can be applied to (6.7b) and we obtain

$$\widetilde{{}^{t}\boldsymbol{u}_b}(\xi) = (p - b)\,\widetilde{{}^{t}\boldsymbol{u}_b}(\xi + \lambda) + \widetilde{\mathbf{w}}_b(\xi)\,\widetilde{{}^{t}\boldsymbol{u}_0}(\xi + \lambda) , \tag{6.10}$$

with ${}^{t}\boldsymbol{u}_0(\xi) = \boldsymbol{s}^T(\xi - \boldsymbol{Y})^{-1}$.

Another type of relation can be obtained by multiply $\widetilde{\boldsymbol{s}}^T(b+\boldsymbol{\Lambda})^{-1}$ on the left hand side of (6.9). We have

$$\begin{aligned} \widetilde{\boldsymbol{s}}^T(b+\boldsymbol{\Lambda})^{-1}\boldsymbol{u}_a(\xi) &= (p-a)\widetilde{\boldsymbol{s}}^T(b+\boldsymbol{\Lambda})^{-1}\widetilde{\boldsymbol{u}}_a(\xi-\lambda) \\ &\quad +\mathbf{v}_a(\xi)\widetilde{\boldsymbol{s}}^T(b+\boldsymbol{\Lambda})^{-1}\widetilde{\boldsymbol{u}}_0(\xi-\lambda) \ , \\ \boldsymbol{s}^T(p+\boldsymbol{\Lambda})(b+\boldsymbol{\Lambda})^{-1}\boldsymbol{u}_a(\xi) &= (p-a)\widetilde{S}_{ab}(\xi-\lambda) + \mathbf{v}_a(\xi)\widetilde{\mathbf{w}}_b(\xi-\lambda) \ , \\ \mathbf{v}_a(\xi)\widetilde{\mathbf{w}}_b(\xi-\lambda) &= 1 + (p-b)S_{ab}(\xi) - (p-a)\widetilde{S}_{ab}(\xi-\lambda) \ . \end{aligned} \tag{6.11}$$

Similarly, multiplying the right hand side of (6.10), we obtain

$$\widetilde{\mathbf{w}}_b(\xi)\mathbf{v}_a(\xi+\lambda) = 1 + (p-b)S_{ab}(\xi+\lambda) - (p-a)\widetilde{S}_{ab}(\xi) \ . \tag{6.12}$$

By proceeding the similar steps, we can derive the relations in other discrete-time directions, namely

$$\mathbf{v}_a(\xi)\widehat{\mathbf{w}}_b(\xi-\mu) = 1 + (q-b)S_{ab}(\xi) - (q-a)\widehat{S}_{ab}(\xi-\mu) \ , \tag{6.13a}$$

$$\mathbf{v}_a(\xi)\overline{\mathbf{w}}_b(\xi-\eta) = 1 + (r-b)S_{ab}(\xi) - (r-a)\overline{S}_{ab}(\xi-\eta) \ , \tag{6.13b}$$

Using the identity

$$\frac{\overline{\widetilde{\mathbf{w}}}_b(\xi-\lambda-\eta)\overline{\mathbf{v}}_a(\xi-\eta)}{\overline{\widehat{\mathbf{w}}}_b(\xi-\mu-\eta)\overline{\mathbf{v}}_a(\xi-\eta)} = \frac{\overline{\widetilde{\mathbf{w}}}_b(\xi-\lambda-\eta)\widetilde{\mathbf{v}}_a(\xi-\lambda)}{\widehat{\widetilde{\mathbf{w}}}_b(\xi-\lambda-\mu)\widetilde{\mathbf{v}}_a(\xi-\lambda)}\,\frac{\widehat{\widetilde{\mathbf{w}}}_b(\xi-\lambda-\mu)\widehat{\mathbf{v}}_a(\xi-\mu)}{\overline{\widehat{\mathbf{w}}}_b(\xi-\mu-\eta)\widehat{\mathbf{v}}_a(\xi-\mu)} \ , \tag{6.14}$$

we can derive

$$\begin{aligned} &\frac{1+(p-b)\overline{S}_{ab}(\xi-\eta)-(p-a)\overline{\widetilde{S}}_{ab}(\xi-\lambda-\eta)}{1+(q-b)\overline{S}_{ab}(\xi-\eta)-(q-a)\overline{\widehat{S}}_{ab}(\xi-\mu-\eta)} \\ &\qquad = \frac{1+(r-b)\widetilde{S}_{ab}(\xi-\lambda)-(r-a)\overline{\widetilde{S}}_{ab}(\xi-\lambda-\eta)}{1+(q-b)\widetilde{S}_{ab}(\xi-\lambda)-(q-a)\widehat{\widetilde{S}}_{ab}(\xi-\lambda-\mu)} \\ &\qquad\qquad \times\frac{1+(p-b)\widehat{S}_{ab}(\xi-\mu)-(p-a)\widehat{\widetilde{S}}_{ab}(\xi-\lambda-\mu)}{1+(r-b)\widehat{S}_{ab}(\xi-\mu)-(r-a)\overline{\widehat{S}}_{ab}(\xi-\mu-\eta)} \ , \end{aligned} \tag{6.15}$$

which is a three-dimensional lattice equation which appeared first (in a slightly different form) in [11]. Effectively, this is the *Schwarzian lattice KP equation* which in its canonical form was first given in [3], cf. also [25].

We now multiply $\widetilde{\boldsymbol{s}}^T$ on the left hand side of (6.9) leading to

$$\begin{aligned} \widetilde{\boldsymbol{s}}^T\boldsymbol{u}_a(\xi) &= (p-a)\widetilde{\boldsymbol{s}}^T\widetilde{\boldsymbol{u}}_a(\xi-\lambda) + \mathbf{v}_a(\xi)\widetilde{\boldsymbol{s}}^T\widetilde{\boldsymbol{u}}_0(\xi-\lambda) \ , \\ \boldsymbol{s}^T(p+\boldsymbol{\Lambda})\boldsymbol{u}_a(\xi) &= (p-a)(1-\widetilde{\mathbf{v}}_a(\xi-\lambda)) + \mathbf{v}_a(\xi)\widetilde{\boldsymbol{s}}^T\widetilde{\boldsymbol{u}}_0(\xi-\lambda) \ . \end{aligned} \tag{6.16}$$

Introducing

$$u_{00}(\xi) = \boldsymbol{s}^T(\xi - \boldsymbol{Y})^{-1}\boldsymbol{r} \ , \tag{6.17}$$

(6.16) can be written in the form

$$(p+\widetilde{u}_{00}(\xi-\lambda))\mathbf{v}_a(\xi) - (p-a)\widetilde{\mathbf{v}}_a(\xi) = a + \boldsymbol{s}^T\boldsymbol{\Lambda}\boldsymbol{u}_a(\xi) \ . \tag{6.18}$$

Another two relations related to the "$\widehat{\ }$" and "$\overline{\ }$" directions can be automatically obtained

$$(q+\widehat{u}_{00}(\xi-\mu))\mathbf{v}_a(\xi) - (q-a)\widehat{\mathbf{v}}_a(\xi-\mu) = a + \boldsymbol{s}^T\boldsymbol{\Lambda}\boldsymbol{u}_a(\xi) \ , \tag{6.19a}$$

$$(r+\overline{u}_{00}(\xi-\eta))\mathbf{v}_a(\xi) - (r-a)\overline{\mathbf{v}}_a(\xi-\eta) = a + \boldsymbol{s}^T\boldsymbol{\Lambda}\boldsymbol{u}_a(\xi) \ . \tag{6.19b}$$

Eliminating the term $\boldsymbol{s}^T\boldsymbol{\Lambda}\boldsymbol{u}_a(\xi)$, we can derive the relations

$$\begin{aligned}(p-q+\widetilde{u}_{00}(\xi-\lambda)-\widehat{u}_{00}(\xi-\mu))\mathbf{v}_a(\xi) &= (p-a)\widetilde{\mathbf{v}}_a(\xi-\lambda)\\ &\quad -(q-a)\widehat{\mathbf{v}}_a(\xi-\mu)\ , \qquad (6.20a)\\ (p-r+\widetilde{u}_{00}(\xi-\lambda)-\overline{u}_{00}(\xi-\eta))\mathbf{v}_a(\xi) &= (p-a)\widetilde{\mathbf{v}}_a(\xi-\lambda)\\ &\quad -(r-a)\overline{\mathbf{v}}_a(\xi-\eta)\ , \qquad (6.20b)\\ (r-q+\overline{u}_{00}(\xi-\eta)-\widehat{u}_{00}(\xi-\mu))\mathbf{v}_a(\xi) &= (r-a)\overline{\mathbf{v}}_a(\xi-\eta)\\ &\quad -(q-a)\widehat{\mathbf{v}}_a(\xi-\mu)\ . \qquad (6.20c)\end{aligned}$$

We now set $p=a$ then (6.20a) and (6.20b) become

$$\begin{aligned}p-q+\widetilde{u}_{00}(\xi-\lambda)-\widehat{u}_{00}(\xi-\mu) &= -(q-p)\frac{\widehat{\mathbf{v}}_p(\xi-\mu)}{\mathbf{v}_p(\xi)}\ , \qquad (6.21a)\\ p-r+\widetilde{u}_{00}(\xi-\lambda)-\overline{u}_{00}(\xi-\eta) &= -(r-p)\frac{\overline{\mathbf{v}}_p(\xi-\eta)}{\mathbf{v}_p(\xi)}\ , \qquad (6.21b)\end{aligned}$$

The combination of (6.21a) and (6.21b) gives

$$\frac{p-q+\widetilde{u}_{00}(\xi-\lambda)-\widehat{u}_{00}(\xi-\mu)}{p-r+\widetilde{u}_{00}(\xi-\lambda)-\overline{u}_{00}(\xi-\eta)}=\frac{p-q+\overline{\widetilde{u}}_{00}(\xi-\lambda-\eta)-\overline{\widehat{u}}_{00}(\xi-\mu-\eta)}{p-r+\widehat{\widetilde{u}}_{00}(\xi-\lambda-\mu)-\widehat{\overline{u}}_{00}(\xi-\eta-\mu)}\ , \qquad (6.22)$$

which is the "*lattice KP equation*", [11], cf. also [10].

From the definition of the function $\mathbf{v}_p(\xi)$ in (6.3), (6.21a) and (6.21b) can be written in terms of the τ-function

$$\begin{aligned}p-q+\widetilde{u}_{00}(\xi-\lambda)-\widehat{u}_{00}(\xi-\mu) &= -(q-p)\frac{\widehat{\widetilde{\tau}}(\xi-\lambda-\mu)}{\widehat{\tau}(\xi-\mu)}\frac{\tau(\xi)}{\widetilde{\tau}(\xi-\lambda)}\ , \qquad (6.23a)\\ p-r+\widetilde{u}_{00}(\xi-\lambda)-\overline{u}_{00}(\xi-\eta) &= -(r-p)\frac{\overline{\widetilde{\tau}}(\xi-\lambda-\eta)}{\overline{\tau}(\xi-\eta)}\frac{\tau(\xi)}{\widetilde{\tau}(\xi-\lambda)}\ . \qquad (6.23b)\end{aligned}$$

From (6.20c), if we set $r=a$ we also have

$$r-q+\overline{u}_{00}(\xi-\lambda)-\widehat{u}_{00}(\xi-\mu)=-(q-r)\frac{\widehat{\overline{\tau}}(\xi-\mu-\eta)}{\widehat{\tau}(\xi-\mu)}\frac{\tau(\xi)}{\overline{\tau}(\xi-\eta)}\ . \qquad (6.24)$$

The combination of (6.23a) (6.23b) (6.24) yields

$$\begin{aligned}&(p-q)\widehat{\widetilde{\tau}}(\xi-\lambda-\mu)\overline{\tau}(\xi-\eta)+(r-p)\overline{\widetilde{\tau}}(\xi-\lambda-\eta)\widehat{\tau}(\xi-\mu)\\ &\qquad +(r-q)\widehat{\overline{\tau}}(\xi-\mu-\eta)\widetilde{\tau}(\xi-\lambda)=0\ , \qquad (6.25)\end{aligned}$$

which is actually the bilinear lattice KP equation, (originally coined DAGTE, cf. [5]).

By introducing the variable τ in (6.1), we derive the linear and nonlinear relations leading to a family of the discrete KP equations (see also [1]). In contrast to the connection established here between the discrete-time RS system and the fully discrete lattice KP equations, the rational CM system treated in [24] is connected to a semi-discrete KP equation which can be obtained by performing a continuum limit on one of the discrete variables. The connection between the RS system and solitons has also been discussed in [18]. In the Section, we present only the discrete versions of the KP equations. For the continuum limit of these equations, it has been studied in [14]. In the next Section we will how this continuum limit corresponds to the non-relativistic limit of the RS system, which after all is viewed as the relativistic variant of the CM system.

7 Non-relativistic limit

In order to perform the non-relativistic limit $\lambda\to 0$, it turns out that the Ansatz for the Lax pair in Section 2 is a bit too restrictive. For that reason, we will discuss here a slight

generalisation of the model by choosing different relativistic parameters for each discrete-time flows. The Lax matrices for the " $\widetilde{\ }$ " direction are given by (2.1b) and the Lax matrices for the " $\widehat{\ }$ " direction are given by (2.16b) with

$$\boldsymbol{K}_0 = \sum_{i,j=1}^{N} \frac{k_i k_j}{x_i - x_j + \mu} E_{ij} \ , \quad \text{and} \quad \boldsymbol{N}_0 = \sum_{i,j=1}^{N} \frac{\widehat{k}_i k_j}{\widehat{x}_i - x_j + \mu} E_{ij} \ , \tag{7.1}$$

where μ is the relativistic parameter corresponding to " $\widehat{\ }$ " direction. Furthermore, we now introduce the third discrete-time flow " $\bar{\ }$ " which the Lax matrices are given by

$$\boldsymbol{F}_0 = \sum_{i,j=1}^{N} \frac{f_i f_j}{x_i - x_j + \eta} E_{ij} \ , \quad \text{and} \quad \boldsymbol{R}_0 = \sum_{i,j=1}^{N} \frac{\overline{f}_i f_j}{\overline{x}_i - x_j + \eta} E_{ij} \ , \tag{7.2}$$

where η is the relativistic parameter corresponding to " $\bar{\ }$ " direction and f_i can be determined by using the same method of what we did for h_i and k_i (see Section 2).

Using the same steps as we did in Section 2, we find the system of discrete equations

$$\frac{p}{\underset{\sim}{p}} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{(x_i - x_j + \lambda)}{(x_i - x_j - \lambda)} = \prod_{j=1}^{N} \frac{(x_i - \widetilde{x}_j)(x_i - \underset{\sim}{x}_j + \lambda)}{(x_i - \underset{\sim}{x}_j)(x_i - \widetilde{x}_j - \lambda)} \ , \tag{7.3a}$$

$$\frac{q}{\underset{\frown}{q}} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{(x_i - x_j + \mu)}{(x_i - x_j - \mu)} = \prod_{j=1}^{N} \frac{(x_i - \widehat{x}_j)(x_i - \underset{\frown}{x}_j + \mu)}{(x_i - \underset{\frown}{x}_j)(x_i - \widehat{x}_j - \mu)} \ , \tag{7.3b}$$

$$\frac{r}{\underline{r}} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{(x_i - x_j + \eta)}{(x_i - x_j - \eta)} = \prod_{j=1}^{N} \frac{(x_i - \overline{x}_j)(x_i - \underline{x}_j + \eta)}{(x_i - \underline{x}_j)(x_i - \overline{x}_j - \eta)} \ , \tag{7.3c}$$

where r is the lattice parameter for the " $\bar{\ }$ " direction and (7.3b), (7.3c) and (7.3c) are the equations of the motion corresponding to *tilde*, *hat* and *bar* directions with their relativistic parameters respectively. The matrix $\boldsymbol{Y}$ of the exact solution (2.25a) now takes the form

$$\begin{aligned} \boldsymbol{Y}(n,m,h) &= (p\boldsymbol{I} + \boldsymbol{\Lambda})^{-n}(q\boldsymbol{I} + \boldsymbol{\Lambda})^{-m}(r\boldsymbol{I} + \boldsymbol{\Lambda})^{-l}\boldsymbol{Y}(0,0)(p\boldsymbol{I} + \boldsymbol{\Lambda})^{n}(q\boldsymbol{I} + \boldsymbol{\Lambda})^{m}(r\boldsymbol{I} + \boldsymbol{\Lambda})^{l} \\ &\quad - np\lambda(p\boldsymbol{I} + \boldsymbol{\Lambda})^{-1} - mq\mu(q\boldsymbol{I} + \boldsymbol{\Lambda})^{-1} - lr\eta(r\boldsymbol{I} + \boldsymbol{\Lambda})^{-1} \ , \end{aligned} \tag{7.4}$$

where l is the discrete-time variable corresponding to the " $\bar{\ }$ " direction.

If we now take the limit on the relativistic parameters such that $n \to \infty$, $p \to \infty$ and $\lambda \to 0$ and we define $\lambda n = -\xi$ and $n/p \to t$. Then (7.4) becomes

$$\begin{aligned} \boldsymbol{Y}(\xi,t,m,h) &= e^{-\boldsymbol{\Lambda}t}(q\boldsymbol{I} + \boldsymbol{\Lambda})^{-m}(r\boldsymbol{I} + \boldsymbol{\Lambda})^{-l}[\boldsymbol{Y}(0,0) + \xi\boldsymbol{I}](q\boldsymbol{I} + \boldsymbol{\Lambda})^{m}(r\boldsymbol{I} + \boldsymbol{\Lambda})^{l}e^{\boldsymbol{\Lambda}t} \\ &\quad - mq\mu(q\boldsymbol{I} + \boldsymbol{\Lambda})^{-1} - lr\eta(r\boldsymbol{I} + \boldsymbol{\Lambda})^{-1} \ , \\ \Rightarrow \boldsymbol{Y}(\xi,m,h) &= (q\boldsymbol{I} + \boldsymbol{\Lambda})^{-m}(r\boldsymbol{I} + \boldsymbol{\Lambda})^{-l}[\boldsymbol{Y}(0,0) + \xi\boldsymbol{I}](q\boldsymbol{I} + \boldsymbol{\Lambda})^{m}(r\boldsymbol{I} + \boldsymbol{\Lambda})^{l} \\ &\quad - mq\mu(q\boldsymbol{I} + \boldsymbol{\Lambda})^{-1} - lr\eta(r\boldsymbol{I} + \boldsymbol{\Lambda})^{-1} \ . \end{aligned} \tag{7.5}$$

(7.5) results from the fact that the exponential can move through annihilating each other and it is a function of one continuous ξ and two discrete-time m, l variables. In fact, the limit $\lambda \to 0$ coincides with what is some time called "*the straight continuum limit*" of the corresponding lattice equation [17].

If we now choose $\mu = 1/q$ and $\eta = 1/r$ (7.5) becomes

$$\begin{aligned} \boldsymbol{Y}(\xi,m,h) &= (q\boldsymbol{I} + \boldsymbol{\Lambda})^{-m}(r\boldsymbol{I} + \boldsymbol{\Lambda})^{-l}[\boldsymbol{Y}(0,0) + \xi\boldsymbol{I}](q\boldsymbol{I} + \boldsymbol{\Lambda})^{m}(r\boldsymbol{I} + \boldsymbol{\Lambda})^{l} - m(q\boldsymbol{I} + \boldsymbol{\Lambda})^{-1} \\ &\quad - l(r\boldsymbol{I} + \boldsymbol{\Lambda})^{-1} \ , \end{aligned} \tag{7.6}$$

which is exactly the exact solution for discrete-time Calogero-Moser system [24, 13].

From the definition of (6.17) together with (A.5), we can express

$$u_{0,0}(\xi) = \boldsymbol{s}^T(\xi - \boldsymbol{Y})^{-1}\boldsymbol{r} = \sum_{i=1}^{N} \frac{h_i^2}{\xi - x_i(n,m)} \ , \tag{7.7}$$

and from the definition of (6.3), we can also express

$$\phi = v_a(\xi) = 1 - \sum_{j=1}^{N} \frac{b_j}{\xi - x_i(n,m)} \ , \tag{7.8}$$

where $\boldsymbol{b} = h^T(a + \boldsymbol{L}_0)^{-1}h$. Eqs (7.7) and (7.8) form the solution for the lattice KP equation (6.22). In the non-relativistic limit we can show that $h_i^2 \to 1$, cf. [15], and hence in this limit Eq. (7.7) becomes

$$u_{0,0}(\xi) \to \sum_{i=1}^{N} \frac{1}{\xi - x_i(n,m)} \ , \tag{7.9}$$

which is the pole-solution for the semi-discrete KP equation as considered in [13, 24].

The conclusion we can draw from this analysis is that the non-relativistic limit performed here provides an explanation as to why the discete-time Calogero-Moser system is derived from the pole reduction of the semi-discrete KP equation, while the Ruijsenaars-Schneider system connects to the fully discrete KP equation, as we have shown in Section 6: In fact, the non-relativistic limit engenders in a natural way a continuum limit in at least one of the discrete variables of the system.

8 Discussion

In this paper we have studied the Lagrangian structure for the Ruijsenaars-Schneider system, and shown that similarly to the Calogero-Moser system, which was treated in [24], it possesses a Lagrangian 1-form structure, both on the discrete-time level as well as in the continuous-time case. Thus, this is the second example of a system of ODEs which exhibits a Lagrangian multi-form structure in the sense of [7] but in a lower-dimensional situation. The present example is important, because in contrast to the CM case where the Lagrane structure is closely related to the Lax representation (and hence inherit the closure relation from the zero-curvature condition), here the relation between the Lax matrices and the Lagrangians are less clear, and the validity of the closure relation is more surprising. Thus, we believe that these results seem to confirm once again that these Lagrangian form structures are fundamental and ubiquitous among integrable systems.

It is well known that the classical RS system is Liouville-integrable in the continuous-time case, [20, 21] and formally so in the discrete-time case [15, 16] as well. However, a note of warning is in order here, pointing to the fact that the discrete-time RS system is fundamentally a complex-valued system of equations, where the solutions may have trajectories which even if the initial values are chosen to be real may wander off in the complex plane in finite time. In such cases, related to the non-selfadjointness of the underlying eigenvalue problems of the Lax pair, it is difficult to make any rigorous statements about the global behavior of the solutions. Nevertheless there are perfectly good reasons to study such systems with non-physical parameters in a complexified phase space, in particular in connection with problems arising from algebraic geometry, cf. e.g. [22]. In the present paper we are interested in these models on a more formal level, where the aim is to establish the emergence of a novel structure (namely that of Lagrangian 1-forms) and to use these models primarily as toy models in order to study how the corresponding variational principles should be implemented.

With regard to the continuous-time model, the Lagrangian 1-form structure in that case had to be established in a rather indirect way, namely by performing systematic limits on Lagrangians of the discrete-time system. We have already pointed out that establishing these

Lagrange structures by Legendre transformation from the known Hamiltonians of the model is not possible, because it is not *a priori* known how these Hamiltonian flows are embedded in a coherent structure, such that we get acceptable Lagrangian components of the 1-form. In this sense the discrete-time model can be viewed as a generating object for such Lagrangians for the continuous-time model. Once those Lagrangians have been found, in fact for a model with unphysical parameters, we expect that it would be perfectly possible to back-engineer the results to find the proper Lagrangian structure for the continuous model in the physical regime of the parameters. We are currently investigating this possibility.

In our view, the importance of this new Lagrangian form structure is that, although a formal proof of this assertion has to be found, it seems clear that this structure is the manifestation of the multidimensional consistency of the equations in the sense of of the papers [12, 2]. It answers the problem of how to find a single Lagrangian framework for a situation where we have a multitude of compatible equations imposed on one and the same (possibly vector-valued) function of many independent variables. In the case of ODEs, which is the case we are dealing with in the present paper, the structure is that of a Lagrangian 1-form describing systems commuting flows in many time-variables (as many as the number of degrees of freedom of the system). The Lagrangians, arising as components of the 1-form, should obviously have a very specific forms, in order for the closure relation to hold subject to the equations of the motion. In fact, such *admissable* Lagrangians are themselves in a sense solutions of the variational principle underlying the structure, namely the one where we consider variations not only with regard to the dependent variables, but with regard to the independent variables as well – in other words where we vary the underlying geometries in the space of independent variables. This forms in our view poses a new paradigm in variational calculus, [4], and possibly a new principle of fundamental physics. That theories which exhibit such structures always correspond to integrable systems (i.e. integrable in other commonly used senses of the term, e.g. through inverse scattering, existence of Lax pairs etc.) is challenging question that remains to be answered.

A The construction of the exact solution

In this Appendix we review the construction of the exact solution for the RS system. The basic relations following from the Lax pair (2.1b) together with the definitions (2.2), lead to

$$\lambda \boldsymbol{M}_0 + \widetilde{\boldsymbol{X}}\boldsymbol{M}_0 - \boldsymbol{M}_0\boldsymbol{X} = \widetilde{h}h^T \ , \tag{A.1}$$

$$\lambda \boldsymbol{M}_0 + \boldsymbol{X}\boldsymbol{L}_0 - \boldsymbol{L}_0\boldsymbol{X} = hh^T \ , \tag{A.2}$$

where we introduce $\boldsymbol{X} = \sum_{i=1}^N x_i E_{ii}$. On the other hand, from the Lax equation (2.2), we obtain the relations

$$\widetilde{\boldsymbol{L}}_0\boldsymbol{M}_0 \quad = \quad \boldsymbol{M}_0\boldsymbol{L}_0 \ , \tag{A.3a}$$

$$\widetilde{\boldsymbol{L}}_0\widetilde{h} - \boldsymbol{M}_0 h \quad = \quad -p\widetilde{h} \ , \tag{A.3b}$$

$$h^T\boldsymbol{L}_0 - \widetilde{h}^T\boldsymbol{M}_0 \quad = \quad -ph^T \ . \tag{A.3c}$$

We now introduce another transformation

$$\boldsymbol{L}_0 = \boldsymbol{U}_0\boldsymbol{\Lambda}\boldsymbol{U}_0^{-1} \ , \qquad \text{and} \qquad \boldsymbol{M}_0 = \widetilde{\boldsymbol{U}}_0\boldsymbol{U}_0^{-1} \ , \tag{A.4}$$

where $\boldsymbol{U}_0$ is an invertible $N \times N$ matrix, and where the matrix $\boldsymbol{\Lambda}$ is constant: $\widetilde{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda}$, as a consequence of (A.3b). Then introducing

$$\boldsymbol{Y} = \boldsymbol{U}_0^{-1}\boldsymbol{X}\boldsymbol{U}_0 \ , \quad \boldsymbol{r} = \boldsymbol{U}_0^{-1} \cdot h \ , \quad \boldsymbol{s}^T = h \cdot \boldsymbol{U}_0 \ , \tag{A.5}$$

we obtain from (A.3) and (A.4),

$$(p\boldsymbol{I} + \boldsymbol{\Lambda}) \cdot \widetilde{\boldsymbol{r}} = \boldsymbol{r} \ , \quad \boldsymbol{s}^T \cdot (\mathrm{p}\boldsymbol{I} + \boldsymbol{\Lambda}) = \widetilde{\boldsymbol{s}}^T \ , \tag{A.6}$$

where $\boldsymbol{I}$ is the unit matrix, as well as from (A.1) and (A.2), we have

$$\begin{aligned} \lambda + \widetilde{\boldsymbol{Y}} - \boldsymbol{Y} &= \widetilde{\boldsymbol{r}}\boldsymbol{s}^T , & \text{(A.7)} \\ \lambda\boldsymbol{\Lambda} + [\boldsymbol{Y}, \boldsymbol{\Lambda}] &= \boldsymbol{r}\boldsymbol{s}^T . & \text{(A.8)} \end{aligned}$$

Eliminating the dyadic $\boldsymbol{r}\boldsymbol{s}^T$ from (A.7) by making use of (A.6), we find the linear equation

$$\widetilde{\boldsymbol{Y}} = (p\boldsymbol{I} + \boldsymbol{\Lambda})^{-1}\boldsymbol{Y}(p\boldsymbol{I} + \boldsymbol{\Lambda}) - \frac{p\lambda}{p\boldsymbol{I} + \boldsymbol{\Lambda}} , \tag{A.9}$$

which can be immediately solved to give

$$\boldsymbol{Y}(n,m) = (p\boldsymbol{I} + \boldsymbol{\Lambda})^{-n}\left(\boldsymbol{Y}(0,m) - \frac{np\lambda}{p\boldsymbol{I} + \boldsymbol{\Lambda}}\right)(p\boldsymbol{I} + \boldsymbol{\Lambda})^{n} , \tag{A.10}$$

subject to the constraint on the initial value matrix

$$[\boldsymbol{Y}(0,m), \boldsymbol{\Lambda}] = \lambda\boldsymbol{\Lambda} + \text{rank } 1 . \tag{A.11}$$

A similar analysis can be applied to create the solution associated with the "$\widehat{\ }$".

Conversely, we can start from a given $N \times N$ diagonal matrix $\boldsymbol{\Lambda}$ with distinct entries, and an initial value matrix $\boldsymbol{Y}(0,0)$ subject to the condition that

$$[\boldsymbol{Y}(0,0), \boldsymbol{\Lambda}] = \lambda\boldsymbol{\Lambda} + rank1 , \tag{A.12}$$

where [,] represents the matrix commutator bracket. Let $\boldsymbol{U}^{-1}(0,0)$ be the matrix that diagonalized $\boldsymbol{Y}(0,0)$, i.e., such that

$$\boldsymbol{Y}(0,0) = \boldsymbol{U}^{-1}(0,0)\,\boldsymbol{X}(0,0)\,\boldsymbol{U}(0,0)) \quad , \quad \boldsymbol{X}(0,0) = \mathrm{diag}(x_1(0,0), \dots, x_N(0,0)) . \tag{A.13}$$

If the eigenvalues of $\boldsymbol{Y}(0,0)$ are distinct (which we can take as an assumption on the initial condition) then $\boldsymbol{U}^{-1}(0,0)$ is determined up to multiplication from the right by a diagonal matrix times a permutation matrix of the columns. (Fixing an ordering of the eigenvalues $x_i(0,0)$, $\boldsymbol{U}^{-1}(0,0)$ unique only up to multiplication by a diagonal matrix from the right). We can fix $\boldsymbol{U}^{-1}(0,0)$ up to an overall multiplicative factor by demanding that

$$[\boldsymbol{Y}(0,0)\,,\,\boldsymbol{\Lambda}] = \lambda\boldsymbol{\Lambda} - \boldsymbol{r}(0,0)\,\boldsymbol{s}^T(0,0) . \tag{A.14}$$

Next, we consider the matrix function given by

$$\begin{aligned} \boldsymbol{Y}(n,m) &= (p\boldsymbol{I} + \boldsymbol{\Lambda})^{-n}(q\boldsymbol{I} + \boldsymbol{\Lambda})^{-m}\boldsymbol{Y}(0,0)(p\boldsymbol{I} + \boldsymbol{\Lambda})^{n}(q\boldsymbol{I} + \boldsymbol{\Lambda})^{m} \\ &\quad -np\lambda(p\boldsymbol{I} + \boldsymbol{\Lambda})^{-1} - mq\lambda(q\boldsymbol{I} + \boldsymbol{\Lambda})^{-1} \end{aligned} \tag{A.15}$$

Let $\boldsymbol{U}(n,m)$ be the matrix diagonalizing $\boldsymbol{Y}(n,m)$ by an appropriate choice of an overall factor (as a function of n and m) this matrix can be fixed such that it obeys:

$$\boldsymbol{r}(n,m) = (p\boldsymbol{I} + \boldsymbol{\Lambda})^{-n}(q\boldsymbol{I} + \boldsymbol{\Lambda})^{-m}\boldsymbol{r}(0,0) \ , \ \text{ and } \ \boldsymbol{s}^T(n,m) = \boldsymbol{s}^T(0,0)(p\boldsymbol{I} + \boldsymbol{\Lambda})(q\boldsymbol{I} + \boldsymbol{\Lambda}) \ , \tag{A.16}$$

and

$$[\boldsymbol{Y}(n,m)\,,\,\boldsymbol{\Lambda}] = \lambda\boldsymbol{\Lambda} - \boldsymbol{r}(n,m)\,\boldsymbol{s}^T(n,m) . \tag{A.17}$$

From the expression (A.15) we can derive the relations

$$\begin{aligned} (p\boldsymbol{I} + \boldsymbol{\Lambda})\,\widetilde{\boldsymbol{Y}} - \boldsymbol{Y}\,(p\boldsymbol{I} + \boldsymbol{\Lambda}) &= -p\lambda , & \text{(A.18a)} \\ (q\boldsymbol{I} + \boldsymbol{\Lambda})\,\widehat{\boldsymbol{Y}} - \boldsymbol{Y}\,(q\boldsymbol{I} + \boldsymbol{\Lambda}) &= -q\lambda, & \text{(A.18b)} \end{aligned}$$

with the usual notation for the shifts in n and m over one unit. Together with the relation (A.17) this subsequently yields:

$$\widetilde{\boldsymbol{Y}} - \boldsymbol{Y} = -p\lambda + \widetilde{\boldsymbol{r}}\boldsymbol{s}^T \quad , \quad \widehat{\boldsymbol{Y}} - \boldsymbol{Y} = -q\lambda + \widehat{\boldsymbol{r}}\boldsymbol{s}^T . \tag{A.19}$$

Reversing these relations by rewriting them in terms of

$$\boldsymbol{X}(n,m) = \boldsymbol{U}(n,m)\,\boldsymbol{Y}(n,m)\,\boldsymbol{U}^{-1}(n,m) \tag{A.20}$$

and now *defining* the Lax matrices by

$$\boldsymbol{L} := \boldsymbol{U}\,\boldsymbol{\Lambda}\,\boldsymbol{U}^{-1} \quad , \quad \boldsymbol{K} := \boldsymbol{U}\,\boldsymbol{\Lambda}\,\boldsymbol{U}^{-1} \ , \tag{A.21}$$

together with

$$\boldsymbol{M} := \widetilde{\boldsymbol{U}}\,\boldsymbol{U}^{-1} \quad , \quad \boldsymbol{N} := \widehat{\boldsymbol{U}}\,\boldsymbol{U}^{-1} \ , \tag{A.22}$$

we recover the relations:

$$[\boldsymbol{X}\,,\,\boldsymbol{L}] = \lambda\boldsymbol{L} + hh^T \ , \tag{A.23a}$$

$$[\boldsymbol{X}\,,\,\boldsymbol{K}] = \lambda\boldsymbol{K} + hh^T \ , \tag{A.23b}$$

and

$$\begin{aligned} \widetilde{\boldsymbol{X}}\,\boldsymbol{M} - \boldsymbol{M}\,\boldsymbol{X} &= -\lambda\boldsymbol{M} + \widetilde{h}h^T \ , && \text{(A.24a)} \\ \widehat{\boldsymbol{X}}\,\boldsymbol{N} - \boldsymbol{N}\,\boldsymbol{X} &= -\lambda\boldsymbol{N} + \widehat{h}h^T \ , && \text{(A.24b)} \end{aligned}$$

which determine the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ as functions of the $x_i(n,m)$ as well as the off-diagonal parts of the matrices $\boldsymbol{L}$ and $\boldsymbol{K}$.

Furthermore, from (A.18) we obtain

$$\begin{aligned} \widetilde{\boldsymbol{L}}\widetilde{\boldsymbol{X}}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{X}\boldsymbol{L} &= -p\widetilde{h}h^T \ , && \text{(A.25a)} \\ \widehat{\boldsymbol{K}}\widehat{\boldsymbol{X}}\boldsymbol{N} - \boldsymbol{N}\boldsymbol{X}\boldsymbol{K} &= -q\widehat{h}h^T \ , && \text{(A.25b)} \end{aligned}$$

which, when combined with the relations of (A.23), yield

$$\begin{aligned} \left(\widetilde{\boldsymbol{L}}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{L}\right)(\boldsymbol{X}+\lambda) + \left(\widetilde{\boldsymbol{L}}\widetilde{h} - \boldsymbol{M}h\right)h^T &= -p\widetilde{h}h^T \ , && \text{(A.26a)} \\ \left(\widehat{\boldsymbol{K}}\boldsymbol{N} - \boldsymbol{N}\boldsymbol{K}\right)(\boldsymbol{X}+\lambda) + \left(\widehat{\boldsymbol{K}}\widehat{h} - \boldsymbol{N}h\right)h^T &= -q\widehat{h}h^T \ . && \text{(A.26b)} \end{aligned}$$

On the other hand, using the relations (A.24) we also obtain

$$\begin{aligned} \left(\widetilde{\boldsymbol{X}}+\lambda\right)\left(\widetilde{\boldsymbol{L}}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{L}\right) + \widetilde{h}\left(h\boldsymbol{L} - h^T\boldsymbol{M}\right) &= -p\widetilde{h}h^T \ , && \text{(A.27a)} \\ \left(\widehat{\boldsymbol{X}}+\lambda\right)\left(\widehat{\boldsymbol{K}}\boldsymbol{N} - \boldsymbol{N}\boldsymbol{K}\right) + \widehat{h}\left(h\boldsymbol{K} - h^T\boldsymbol{N}\right) &= -q\widehat{h}h^T \ . && \text{(A.27b)} \end{aligned}$$

From the relations (A.26) and (A.27) it follows that the Lax equations hold and their form is determined up to the diagonal part of the matrices $\boldsymbol{L}$ and $\boldsymbol{K}$.

B The compatibility between the Lax matrices

In this Appendix we will show the explicit computation of the compatibility between Lax matrices.

B.1 The compatibility between $\boldsymbol{L}_\kappa$ and $\boldsymbol{K}_\kappa$

Consider the relation

$$\boldsymbol{L}_\kappa\boldsymbol{K}_\kappa = \boldsymbol{K}_\kappa\boldsymbol{L}_\kappa \ , \tag{B.1}$$

yielding

$$\begin{aligned} & h_i^2\sum_{j=1}^{N}k_j^2\left(\frac{1}{\kappa} - \frac{1}{\kappa+2\lambda+x_i-x_l} + \frac{2}{x_i-x_j+\lambda}\right) \\ &\qquad = k_i^2\sum_{j=1}^{N}h_j^2\left(\frac{1}{\kappa} - \frac{1}{\kappa+2\lambda+x_i-x_l} + \frac{2}{x_i-x_j+\lambda}\right) \ . \end{aligned} \tag{B.2}$$

(B.2) leads to the condition

$$h_i^2 \sum_{j=1}^{N} k_j^2 = k_i^2 \sum_{j=1}^{N} h_j^2 \ . \tag{B.3}$$

Introducing $\beta = \sum_{j=1}^N h_j^2 / \sum_{j=1}^N k_j^2$, we have

$$h_i^2 = \beta k_i^2 \ . \tag{B.4}$$

B.2 The compatibility between $\boldsymbol{L}_\kappa$ and $\boldsymbol{N}_\kappa$

consider the relation

$$\widehat{\boldsymbol{L}}_\kappa \boldsymbol{N}_\kappa = \boldsymbol{N}_\kappa \boldsymbol{L}_\kappa \ , \tag{B.5}$$

yielding

$$\begin{aligned} &\widehat{h}_i^2 \sum_{j=1}^{N} \widehat{k}_j^2 \left(\frac{1}{\kappa} - \frac{1}{\kappa + 2\lambda + \widehat{x}_i - x_l} + \frac{1}{\widehat{x}_i - \widehat{x}_j + \lambda} + \frac{1}{\widehat{x}_i - x_j + \lambda} \right) \\ &\quad = \widehat{k}_i^2 \sum_{j=1}^{N} h_j^2 \left(\frac{1}{\kappa} - \frac{1}{\kappa + 2\lambda + \widehat{x}_i - x_l} + \frac{1}{\widehat{x}_i - x_j + \lambda} + \frac{1}{x_i - x_j + \lambda} \right) \ . \end{aligned} \tag{B.6}$$

The terms with the κ lead to the condition

$$\widehat{h}_i^2 = \alpha \widehat{k}_i^2 \ , \tag{B.7}$$

where $\alpha = \sum_{j=1}^N h_j^2 / \sum_{j=1}^N \widehat{k}_j^2$, we have

$$h_i^2 = \underset{\frown}{\alpha} k_i^2 \ . \tag{B.8}$$

Comparing (B.8) with (B.4), we have $\underset{\frown}{\alpha} = \beta$ which implies $\sum_{j=1}^N h_j^2 = \sum_{j=1}^N \widehat{h}_j^2$.

The remaining terms in (B.6) lead to the identity

$$\sum_{j=1}^{N} \left(\frac{\alpha \widehat{k}_j^2}{\widehat{x}_i - \widehat{x}_j + \lambda} - \frac{\underset{\frown}{\alpha} k_j^2}{\widehat{x}_i - x_j + \lambda} \right) = \sum_{j=1}^{N} \left(\frac{\underset{\frown}{\alpha} k_j^2}{\widehat{x}_i - x_l + \lambda} - \frac{\alpha \widehat{k}_j^2}{\widehat{x}_i - x_l + \lambda} \right) \ . \tag{B.9}$$

Using the Lagrange interpolation formula, we have

$$\alpha \widehat{k}_i^2 = \frac{\prod_{j=1}^N (\widehat{x}_i - \widehat{x}_j - \lambda)(\widehat{x}_i - x_j + \lambda)}{\prod_{j \neq i}^N (\widehat{x}_i - \widehat{x}_j) \prod_{j=1}^N (\widehat{x}_i - x_j)} \ , \tag{B.10}$$

$$\underset{\frown}{\alpha} k_i^2 = \frac{\prod_{j=1}^N (x_i - \widehat{x}_j - \lambda)(x_i - x_j + \lambda)}{\prod_{j \neq i}^N (x_i - x_j) \prod_{j=1}^N (x_i - \widehat{x}_j)} \ , \tag{B.11}$$

from which we obtain the equations of motion (2.19).

B.3 The compatibility between $\boldsymbol{K}_\kappa$ and $\boldsymbol{M}_\kappa$

consider the relation

$$\widetilde{\boldsymbol{K}}_\kappa \boldsymbol{M}_\kappa = \boldsymbol{M}_\kappa \boldsymbol{K}_\kappa \ , \tag{B.12}$$

yielding

$$\begin{aligned} &\widetilde{h}_i^2 \sum_{j=1}^{N} \widetilde{k}_j^2 \left(\frac{1}{\kappa} - \frac{1}{\kappa + 2\lambda + \widetilde{x}_i - x_l} + \frac{1}{\widetilde{x}_i - \widetilde{x}_j + \lambda} + \frac{1}{\widetilde{x}_i - x_j + \lambda} \right) \\ &\quad = \widetilde{k}_i^2 \sum_{j=1}^{N} h_j^2 \left(\frac{1}{\kappa} - \frac{1}{\kappa + 2\lambda + \widetilde{x}_i - x_l} + \frac{1}{\widetilde{x}_i - x_j + \lambda} + \frac{1}{x_i - x_j + \lambda} \right) \ . \end{aligned} \tag{B.13}$$

The terms with the κ lead to the condition

$$\widetilde{h}_i^2 = \mu \widetilde{k}_i^2 \ , \tag{B.14}$$

where $\mu = \sum_{j=1}^{N} h_j^2 / \sum_{j=1}^{N} \widetilde{k}_j^2$, we have

$$h_i^2 = \underset{\sim}{\mu} k_i^2 \ . \tag{B.15}$$

Comparing (B.15) with (B.4), we have $\underset{\sim}{\mu} = \beta$ which implies $\sum_{j=1}^{N} k_j^2 = \sum_{j=1}^{N} \widetilde{k}_j^2$.

The remaining terms in (B.6) lead to the identity

$$\sum_{j=1}^{N} \left(\frac{\mu \widetilde{k}_j^2}{\widetilde{x}_i - \widetilde{x}_j + \lambda} - \frac{\underset{\sim}{\mu} k_j^2}{\widetilde{x}_i - x_j + \lambda} \right) = \sum_{j=1}^{N} \left(\frac{\underset{\sim}{\mu} k_j^2}{\widetilde{x}_i - x_l + \lambda} - \frac{\mu \widetilde{k}_j^2}{\widetilde{x}_i - x_l + \lambda} \right) \ . \tag{B.16}$$

Using the Lagrange interpolation formula, we have

$$\mu \widetilde{k}_i^2 \quad = \quad \frac{\prod_{j=1}^{N} (\widetilde{x}_i - \widetilde{x}_j - \lambda)(\widetilde{x}_i - x_j + \lambda)}{\prod_{j \neq i}^{N} (\widetilde{x}_i - \widetilde{x}_j) \prod_{j=1}^{N} (\widetilde{x}_i - x_j)} \ , \tag{B.17}$$

$$\underset{\sim}{\mu} k_i^2 \quad = \quad \frac{\prod_{j=1}^{N} (x_i - \widetilde{x}_j - \lambda)(x_i - x_j + \lambda)}{\prod_{j \neq i}^{N} (x_i - x_j) \prod_{j=1}^{N} (x_i - \widetilde{x}_j)} \ , \tag{B.18}$$

from which we obtain the equations of motion (2.12).

C Examples

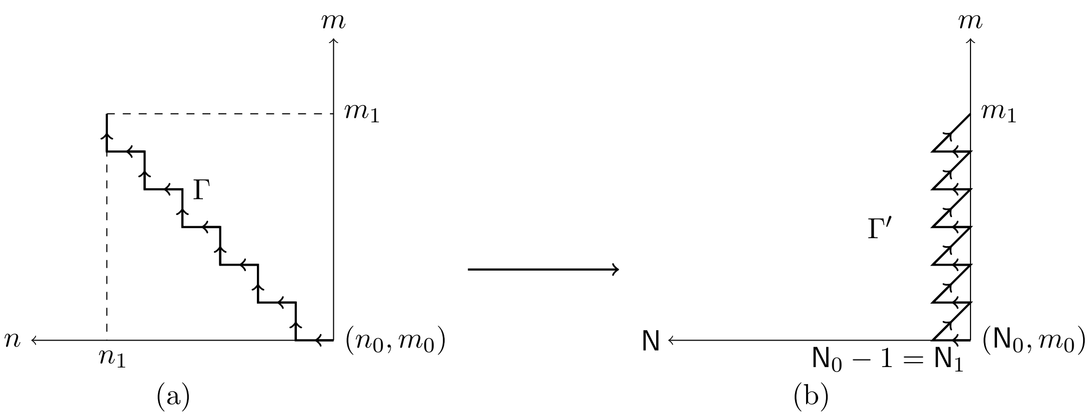

Figure 2: The effect of changing variables on the discrete curve I.

In this Appendix, we will investigate how to derive the discrete Euler-Lagrange equation from the variational principle. For general discrete curves it is cumbersome to implement the variational principle because of the notation it would require. We will, however, demonstrate how the principle works for a few simple cases: **(a)** the curve shown in Fig. (2), **(b)** the curve shown in Fig. (3).

Case(a): The curve shown in Fig. (2): We now introduce a new variable $\mathsf{N} = n + m$ (which will play an important role in the next section) together with the change of notation

$$\boldsymbol{x}(n, m) \mapsto \boldsymbol{x}(\mathsf{N}, m), \qquad \widetilde{\boldsymbol{x}} := \boldsymbol{x}(\mathsf{N}+1, m) \quad \text{and} \quad \widehat{\boldsymbol{x}} := \boldsymbol{x}(\mathsf{N}, m+1) \ ,$$

and so we work with the curve given in Fig. (2b). The action evaluated on this curve can be written in the form

$$S[\boldsymbol{x};\Gamma'] = \sum_{m=m_0}^{m_1-1} -\mathscr{L}_{(\mathsf{N})}(\boldsymbol{x}(\mathsf{N}_0-1,m),\boldsymbol{x}(\mathsf{N}_0,m)) + \sum_{m=m_0}^{m_1-1} \mathscr{L}_{(m)}(\boldsymbol{x}(\mathsf{N}_0-1,m),\boldsymbol{x}(\mathsf{N}_0,m+1)) \quad \text{(C.1)}$$

where

$$\begin{aligned}
\mathscr{L}_{(\mathsf{N})}(\boldsymbol{x},\boldsymbol{y}) &= \sum_{i,j=1}^{N}\left(f(y_i-x_j)-f(y_i-x_j-\lambda)\right)-\frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N} f(y_i-y_j+\lambda)\\
&\quad -\frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N} f(x_i-x_j+\lambda)-\ln\left|\frac{p}{\sqrt{\beta}}\right|\sum_{i=1}^{N}(y_i-x_i)\;, \qquad \text{(C.2)}\\
\mathscr{L}_{(m)}(\boldsymbol{x},\boldsymbol{y}) &= \sum_{i,j=1}^{N}\left(f(x_i-y_j)-f(x_i-y_j-\lambda)\right)-\frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N} f(x_i-x_j+\lambda)\\
&\quad -\frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N} f(y_i-y_j+\lambda)-\ln\left|q\sqrt{\beta}\right|\sum_{i=1}^{N}(x_i-y_i)\;, \qquad \text{(C.3)}
\end{aligned}$$

The minus sign in (C.1) indicates the reverse direction of the Lagrangian $\mathsf{L}_{(\mathsf{N})}$ along the horizontal links. Performing the variation $\boldsymbol{x}\mapsto\boldsymbol{x}+\delta\boldsymbol{x}$, we have

$$\begin{aligned}
&\delta S = 0 = \\
&\sum_{m=m_0}^{m_1-1}\Bigg(-\frac{\partial\mathscr{L}_{(\mathsf{N})}(\boldsymbol{x}(\mathsf{N}_0-1,m),\boldsymbol{x}(\mathsf{N}_0,m))}{\partial\boldsymbol{x}(\mathsf{N}_0,m)}\delta\boldsymbol{x}(\mathsf{N}_0,m)\\
&\qquad -\frac{\partial\mathscr{L}_{(\mathsf{N})}(\boldsymbol{x}(\mathsf{N}_0-1,m),\boldsymbol{x}(\mathsf{N}_0,m))}{\partial\boldsymbol{x}(\mathsf{N}_0-1,m)}\delta\boldsymbol{x}(\mathsf{N}_0-1,m)\Bigg)\\
&+\sum_{m=m_0}^{m_1-1}\Bigg(\frac{\partial\mathscr{L}_{(m)}(\boldsymbol{x}(\mathsf{N}_0-1,m),\boldsymbol{x}(\mathsf{N}_0,m+1))}{\partial\boldsymbol{x}(\mathsf{N}_0,m+1)}\delta\boldsymbol{x}(\mathsf{N}_0,m+1)\\
&\qquad +\frac{\partial\mathscr{L}_{(m)}(\boldsymbol{x}(\mathsf{N}_0-1,m),\boldsymbol{x}(\mathsf{N}_0,m+1))}{\partial\boldsymbol{x}(\mathsf{N}_0-1,m)}\delta\boldsymbol{x}(\mathsf{N}_0-1,m)\Bigg)\\
&. \qquad \text{(C.4)}
\end{aligned}$$

We now obtain the Euler-Lagrange equations

$$-\frac{\partial\mathscr{L}_{(\mathsf{N})}(\boldsymbol{x}(\mathsf{N}_0-1,m),\boldsymbol{x}(\mathsf{N}_0,m))}{\partial\boldsymbol{x}(\mathsf{N}_0,m)}+\frac{\partial\mathscr{L}_{(m)}(\boldsymbol{x}(\mathsf{N}_0-1,m-1),\boldsymbol{x}(\mathsf{N}_0,m)}{\partial\boldsymbol{x}(\mathsf{N}_0,m)}=0\;, \qquad \text{(C.5a)}$$

$$-\frac{\partial\mathscr{L}_{(\mathsf{N})}(\boldsymbol{x}(\mathsf{N}_0-1,m),\boldsymbol{x}(\mathsf{N}_0,m))}{\partial\boldsymbol{x}(\mathsf{N}_0-1,m)}+\frac{\partial\mathscr{L}_{(m)}(\boldsymbol{x}(\mathsf{N}_0-1,m),\boldsymbol{x}(\mathsf{N}_0,m+1)}{\partial\boldsymbol{x}(\mathsf{N}_0-1,m)}=0\;, \qquad \text{(C.5b)}$$

which produce

$$\begin{aligned}
\ln\left|\frac{p}{q\beta}\right| &= \sum_{j=1}^{N}\Bigg(\ln\left|\frac{\boldsymbol{x}_i(\mathsf{N}_0,m)-\boldsymbol{x}_j(\mathsf{N}_0-1,m)}{\boldsymbol{x}_i(\mathsf{N}_0,m)-\boldsymbol{x}_j(\mathsf{N}_0-1,m)+\lambda}\right|\\
&\qquad +\left|\frac{\boldsymbol{x}_i(\mathsf{N}_0,m)-\boldsymbol{x}_j(\mathsf{N}_0-1,m-1)+\lambda}{\boldsymbol{x}_i(\mathsf{N}_0,m)-\boldsymbol{x}_j(\mathsf{N}_0-1,m-1)}\right|\Bigg)\;,\\
\ln\left|\frac{p}{q\beta}\right| &= \sum_{j=1}^{N}\Bigg(\ln\left|\frac{\boldsymbol{x}_i(\mathsf{N}_0-1,m)-\boldsymbol{x}_j(\mathsf{N}_0,m+1)}{\boldsymbol{x}_i(\mathsf{N}_0-1,m)-\boldsymbol{x}_j(\mathsf{N}_0,m+1)-\lambda}\right|\\
&\qquad +\left|\frac{\boldsymbol{x}_i(\mathsf{N}_0-1,m)-\boldsymbol{x}_j(\mathsf{N}_0,m)-\lambda}{\boldsymbol{x}_i(\mathsf{N}_0-1,m)-\boldsymbol{x}_j(\mathsf{N}_0,m)}\right|\Bigg)\;,
\end{aligned}$$

which are equivalent to (2.24a) and (2.24b), respectively.

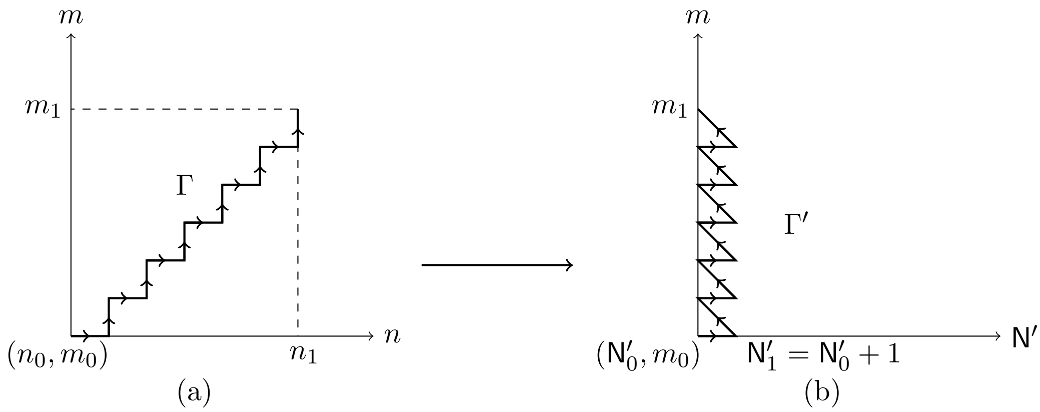

Figure 3: The effect of changing variables on the discrete curve II.

Case(b): The curve shown in Fig. (3): Introducing the variable $\mathsf{N}' = n - m$, the corresponding curve is given in Fig. (3b). The action evaluated on the curve Γ' reads

$$S[\boldsymbol{x};\Gamma'] = \sum_{m=m_0}^{m_1-1} \mathscr{L}_{(\mathsf{N}')}(\boldsymbol{x}(\mathsf{N}'_0,m),\boldsymbol{x}(\mathsf{N}'_0+1,m)) + \sum_{m=m_0}^{m_1-1} \mathscr{L}_{(m)}(\boldsymbol{x}(\mathsf{N}'_0+1,m),\boldsymbol{x}(\mathsf{N}'_0,m+1)), \tag{C.7}$$

where

$$\mathscr{L}_{(\mathsf{N}')}(\boldsymbol{x},\boldsymbol{y}) = \sum_{i,j=1}^{N} \left(f(x_i-y_j)-f(x_i-y_j-\lambda)\right) - \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N} f(y_i-y_j+\lambda) - \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N} f(x_i-x_j+\lambda) - \ln\left|\frac{p}{\sqrt{\beta}}\right|\sum_{i=1}^{N}(x_i-y_i)\ , \tag{C.8}$$

$$\mathscr{L}_{(m)}(\boldsymbol{x},\boldsymbol{y}) = \sum_{i,j=1}^{N} \left(f(x_i-y_j)-f(x_i-y_j-\lambda)\right) - \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N} f(x_i-x_j+\lambda) - \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N} f(y_i-y_j+\lambda) - \ln\left|q\sqrt{\beta}\right|\sum_{i=1}^{N}(x_i-y_i)\ , \tag{C.9}$$

Performing the variation $\boldsymbol{x} \mapsto \boldsymbol{x} + \delta\boldsymbol{x}$, we have

$$
\begin{aligned}
&\delta S = 0 = \\
&\sum_{m=m_0}^{m_1-1}\Bigg(\frac{\partial\mathscr{L}_{(\mathsf{N}')}(\boldsymbol{x}(\mathsf{N}_0',m),\boldsymbol{x}(\mathsf{N}_0'+1,m)}{\partial\boldsymbol{x}(\mathsf{N}_0',m)}\delta\boldsymbol{x}(\mathsf{N}_0',m) \\
&\qquad\qquad\qquad + \frac{\partial\mathscr{L}_{(\mathsf{N}')}(\boldsymbol{x}(\mathsf{N}_0',m),\boldsymbol{x}(\mathsf{N}_0'+1,m)}{\partial\boldsymbol{x}(\mathsf{N}_0'+1,m)}\delta\boldsymbol{x}(\mathsf{N}_0'+1,m)\Bigg) \\
&+\sum_{m=m_0}^{m_1-1}\Bigg(\frac{\partial\mathscr{L}_{(m)}(\boldsymbol{x}(\mathsf{N}_0'+1,m),\boldsymbol{x}(\mathsf{N}_0',m+1)}{\partial\boldsymbol{x}(\mathsf{N}_0',m+1)}\delta\boldsymbol{x}(\mathsf{N}_0',m+1) \\
&\qquad\qquad\qquad +\frac{\partial\mathscr{L}_{(m)}(\boldsymbol{x}(\mathsf{N}_0'+1,m),\boldsymbol{x}(\mathsf{N}_0',m+1)}{\partial\boldsymbol{x}(\mathsf{N}_0'+1,m)}\delta\boldsymbol{x}(\mathsf{N}_0'+1,m)\Bigg)\ .
\end{aligned}
\tag{C.10}
$$

We now obtain the Euler-Lagrange equations

$$
\frac{\partial\mathscr{L}_{(\mathsf{N}')}(\boldsymbol{x}(\mathsf{N}_0',m),\boldsymbol{x}(\mathsf{N}_0'+1,m)}{\partial\boldsymbol{x}(\mathsf{N}_0',m)} + \frac{\partial\mathscr{L}_{(m)}(\boldsymbol{x}(\mathsf{N}_0'+1,m-1),\boldsymbol{x}(\mathsf{N}_0',m)}{\partial\boldsymbol{x}(\mathsf{N}_0',m+1)} = 0\ , \tag{C.11a}
$$

$$
\frac{\partial\mathscr{L}_{(\mathsf{N}')}(\boldsymbol{x}(\mathsf{N}_0',m),\boldsymbol{x}(\mathsf{N}_0'+1,m)}{\partial\boldsymbol{x}(\mathsf{N}_0'+1,m)} + \frac{\partial\mathscr{L}_{(m)}(\boldsymbol{x}(\mathsf{N}_0'+1,m),\boldsymbol{x}(\mathsf{N}_0',m+1)}{\partial\boldsymbol{x}(\mathsf{N}_0'+1,m)} = 0\ , \tag{C.11b}
$$

which produce

$$
\begin{aligned}
\ln\left|\frac{p}{q\beta}\right| &= \sum_{j=1}^{N}\Bigg(\ln\left|\frac{\boldsymbol{x}_i(\mathsf{N}_0',m)-\boldsymbol{x}_j(\mathsf{N}_0'-1,m)}{\boldsymbol{x}_i(\mathsf{N}_0',m)-\boldsymbol{x}_j(\mathsf{N}_0'-1,m)+\lambda}\right| \\
&\qquad\qquad + \ln\left|\frac{\boldsymbol{x}_i(\mathsf{N}_0',m)-\boldsymbol{x}_j(\mathsf{N}_0'-1,m-1)+\lambda}{\boldsymbol{x}_i(\mathsf{N}_0',m)-\boldsymbol{x}_j(\mathsf{N}_0'-1,m-1)}\right|\Bigg)\ , \\
\ln\left|\frac{p}{q\beta}\right| &= \sum_{j=1}^{N}\Bigg(\ln\left|\frac{\boldsymbol{x}_i(\mathsf{N}_0'-1,m)-\boldsymbol{x}_j(\mathsf{N}_0',m+1)}{\boldsymbol{x}_i(\mathsf{N}_0'-1,m)-\boldsymbol{x}_j(\mathsf{N}_0',m+1)-\lambda}\right| \\
&\qquad\qquad + \ln\left|\frac{\boldsymbol{x}_i(\mathsf{N}_0'-1,m)-\boldsymbol{x}_j(\mathsf{N}_0',m)-\lambda}{\boldsymbol{x}_i(\mathsf{N}_0'-1,m)-\boldsymbol{x}_j(\mathsf{N}_0',m)}\right|\Bigg)\ ,
\end{aligned}
$$

which are equivalent to (2.24a) and (2.24b), respectively.

By working with the specific type of curves given in Fig. (2) and Fig. (3), we can perform the variational principle with either the new variables (N, m) or (N', m). Even though we have these two possibilities, the natural choice is to work with (N, m). This is because the variable N' leads to difficulties in performing the skew limit in the next section.

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